Phys 4247 - Computational Project Solutions

1. Distances

Start w/ the Friedmann equation including matter, radiation and a Cosmological constant:

$$\dot{a}^2 = \frac{8\pi G}{3}\left(\rho_{matt} + \rho_{rad}\right)a^2 + \frac{\Lambda a^2}{3} \qquad (K=0)$$

$$= \frac{8\pi G}{3}\rho_{matt,0}\frac{a_0^3}{a^3}a^2 + \frac{8\pi G}{3}\rho_{rad,0}\frac{a_0^4}{a^4}a^2 + \frac{\Lambda a^2}{3}\frac{H_0^2}{H_0^2}$$

$$= \frac{8\pi G}{3}\rho_{matt,0}(1+z)^3 a^2\frac{H_0^2}{H_0^2} + \frac{8\pi G}{3}\rho_{rad,0}(1+z)^4 a^2\frac{H_0^2}{H_0^2} + \Omega_{\Lambda,0}H_0^2 a^2$$

since $(1+z) = \frac{a_0}{a}$ and $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$

$$\therefore \frac{\dot{a}^2}{a^2} = \Omega_{m,0}(1+z)^3 H_0^2 + \Omega_{rad,0}(1+z)^4 H_0^2 + \Omega_{\Lambda,0}H_0^2$$

This last equation uses the definitions $\Omega_{m,0} = \frac{\rho_{matt,0}}{\rho_{c,0}}$ and $\Omega_{rad,0} = \frac{\rho_{rad,0}}{\rho_{c,0}}$ where $\rho_{c,0} = \frac{3H_0^2}{8\pi G}$.

To find d_{lum}, d_{ang} & d_{prop} need to compute $r_0 = a_0\int_0^r \frac{dr}{\sqrt{1-Kr^2}} = ca_0\int_{t_i}^{t_0}\frac{c\,dt}{a(t)}$ where the last equality comes from setting $ds^2=0$ in the RW metric. Want to change the integral to one in terms of z, so use the following coordinate transformations: $\frac{da}{dt} = \dot{a} \rightarrow dt = \frac{da}{\dot{a}}$

and since $(1+z) = \frac{a_0}{a} \rightarrow z = \frac{a_0}{a} - 1 \rightarrow \frac{dz}{da} = -\frac{a_0}{a^2} = -\frac{(1+z)}{a}$

$$\therefore da = -\frac{a\,dz}{(1+z)} \rightarrow dt = -\frac{a\,dz}{\dot{a}(1+z)}$$

So, integral becomes $r_0 = ca_0 \int_z^0 \frac{-a\,dz}{\dot{a}(1+z)a} = c\int_0^z \frac{dz}{(1+z)}\frac{aa_0}{\dot{a}a} = c\int_0^z dz\left(\frac{a}{\dot{a}}\right)$

$$r_0 = c\int_0^z \frac{dz}{H_0\left[\Omega_{m,0}(1+z)^3 + \Omega_{rad,0}(1+z)^4 + \Omega_{\Lambda,0}\right]^{1/2}}$$

Then $d_{prop} = r_0$, $d_{lum} = r_0(1+z)$ and $d_{ang} = \frac{r_0}{(1+z)}$.

2. Age

For this need to re-write the left-hand side of the Friedmann equation in terms of z and t, so use

$$\frac{dz}{dt} = -\frac{a_0}{a^2}\frac{da}{dt} \rightarrow \frac{da}{dt} = -\frac{a^2}{a_0}\frac{dz}{dt}$$

$$\therefore \dot{a}^2 = \frac{a^{4\,2}}{a_0^2}\left(\frac{dz}{dt}\right)^2 = a^2 H_0^2\left[\Omega_{m,0}(1+z)^3 + \Omega_{rad,0}(1+z)^4 + \Omega_{\Lambda,0}\right]$$

$$\frac{a^2}{a_0^2}\left(\frac{dz}{dt}\right) = \frac{1}{(1+z)^2}\left(\frac{dz}{dt}\right) = H_0^2\left[\Omega_{m,0}(1+z)^3 + \Omega_{rad,0}(1+z)^4 + \Omega_{\Lambda,0}\right]$$

$$\frac{1}{(1+z)}\frac{dz}{dt} = H_0\left[\Omega_{m,0}(1+z)^3 + \Omega_{rad,0}(1+z)^4 + \Omega_{\Lambda,0}\right]^{1/2}$$

$$\therefore H_0\int_0^t dt = H_0 t = \int_{+\infty}^z \frac{dz}{(1+z)\left[\Omega_{m,0}(1+z)^3 + \Omega_{rad,0}(1+z)^4 + \Omega_{\Lambda,0}\right]^{1/2}}$$ where t is the age at redshift z.

To convert to Gyr, use $H_0 = \left(\frac{70\ \text{Km}}{\text{s Mpc}}\right)\cdot\left(1.023\times10^{-3}\right)$

to convert H_0 into units of Gyr^{-1}!

Finally, on the age plot, the redshift of decoupling is $z = 1130$ and the redshift of recombination is $z = 1370$. To find the redshift of matter-Λ equality, compute the scale factor when their densities are equal:

$$\frac{\Lambda}{8\pi G} = \rho_{matt,0}\left(\frac{a_0}{a_{m\Lambda}}\right)^3$$

$$\frac{\Lambda 3H_0^2}{8\pi G 3H_0^2} = \rho_{matt,0}\left(\frac{a_0}{a_{m\Lambda}}\right)^3$$

$$\Omega_{\Lambda,0}\frac{3H_0^2}{8\pi G} = \rho_{matt,0}\left(\frac{a_0}{a_{m\Lambda}}\right)^3$$

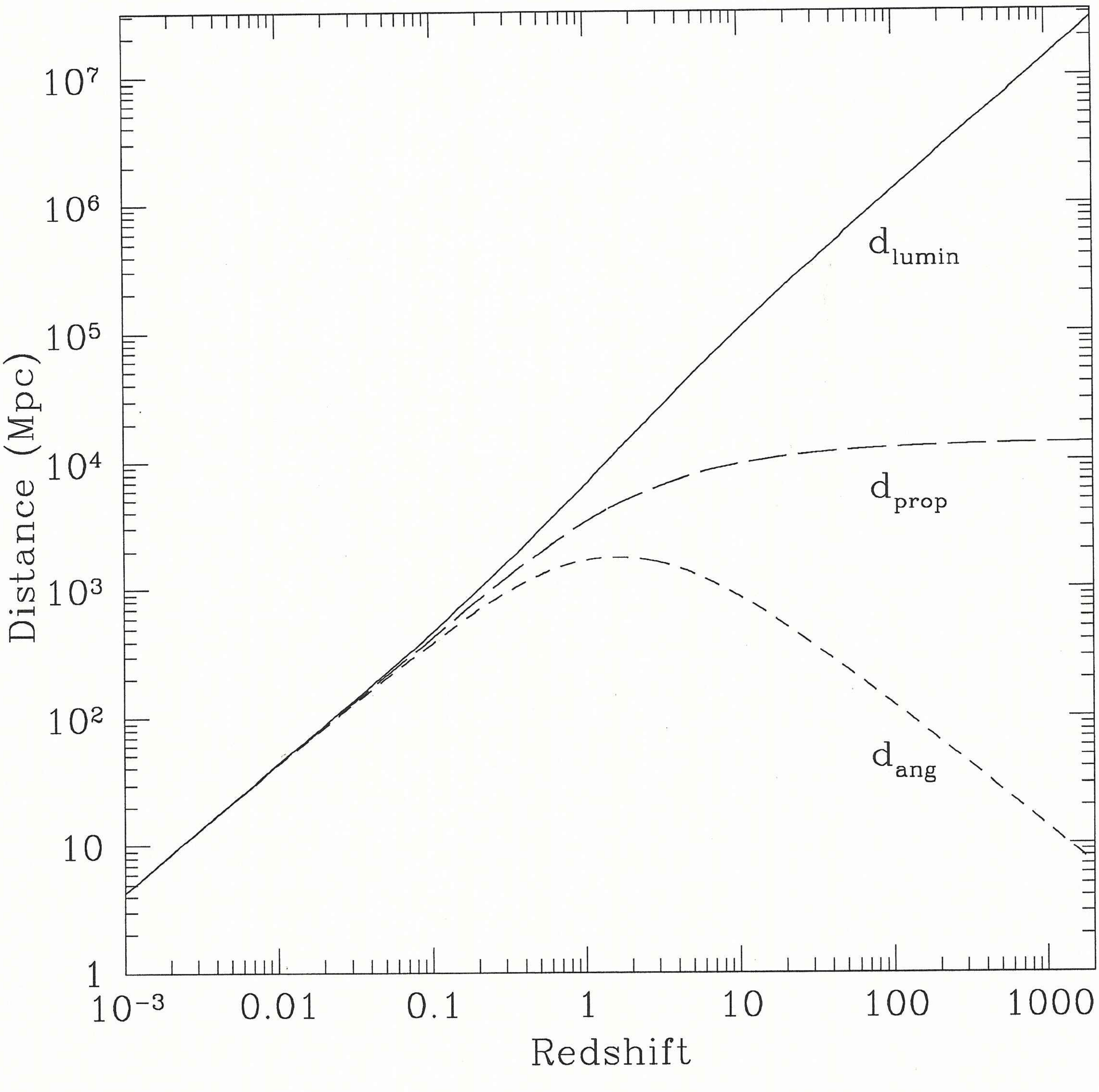
$$\Omega_{\Lambda,0}\rho_{c,0} = \rho_{matt,0}\left(\frac{a_0}{a_{m\Lambda}}\right)^3$$

$$\Omega_{\Lambda,0} = \Omega_{m,0}\left(\frac{a_0}{a_{m\Lambda}}\right)^3$$

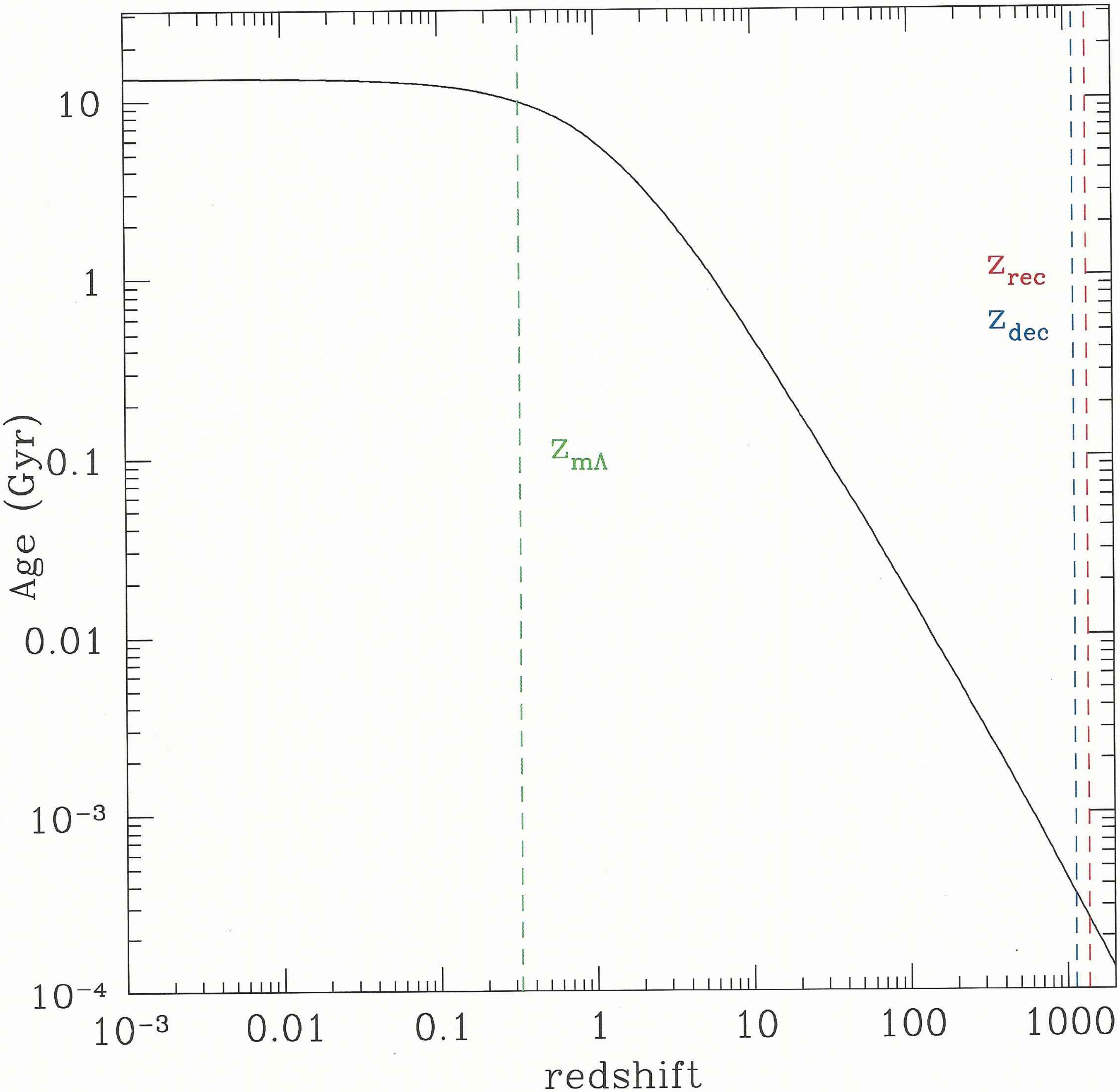
$$\left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{1/3} = \frac{a_0}{a_{m\Lambda}} \rightarrow a_{m\Lambda} = a_0\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}$$

if $\Omega_{m,0} = 0.3$; $\Omega_{\Lambda,0} = 0.7$; $a_0 = 1$ $\rightarrow a_{m\Lambda} = 0.754$

$$\rightarrow z_{m\Lambda} = \frac{1}{0.754} - 1 = 0.326$$



Distance (Mpc)
1
10
10²
10³
10⁴
10⁵
10⁶
10⁷
d_lumin
d_prop
d_ang
10⁻³
0.01
0.1
1
10
100
1000
Redshift



10
1
0.1
0.01
10^{-3}
10^{-4}
Age (Gyr)
10^{-3}
0.01
0.1
1
10
100
1000
redshift
$z_{m\Lambda}$
z_{rec}
z_{dec}