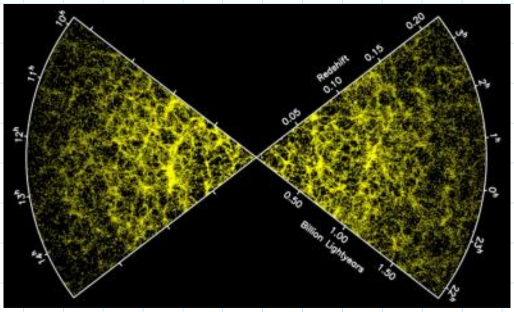
The Formation of Structure



Clearly, on scales ≤ 100 Mpc, the Univ. is not homogeneous - there are superclusters, clusters, galaxies, voids, etc.

Overall topology is similar to a sponge. The basic mechanism for growing the large structures (voids, superclusters) is gravitational instability.

Inflation predicts slight fluctuations in the density of the Univ. They are observed on the CMB. These will grow due to gravity. If fluctuations are small, we can predict their evolution.

Grav. Instability in a homogeneous, flat, non-expanding Univ.

At some time t, the spatially-averaged density is $\bar{\rho}(t) = \frac{1}{V}\int \rho(\vec{r},t)\,dV$ (V=volume)

and $V \gg$ than the biggest structure of the Univ.

Define a dimensionless density fluctuation

$$\delta(\vec{r},t) \equiv \frac{\rho(\vec{r},t) - \bar{\rho}(t)}{\bar{\rho}(t)}$$

$\delta < 0$ in underdense region

$\delta > 0$ in overdense region

In our discussions, $|\delta| \ll 1$ and use linear perturbation theory.

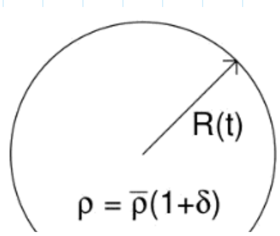
e.g. Consider adding a small amount of mass to a sphere of radius R, so that the density within is $\bar{\rho}(1+\delta)$ where $\delta \ll 1$

If δ is uniform within the sphere, then the acc'n at the surface due to the excess mass is

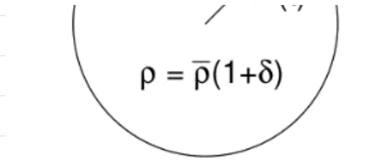
$$\ddot{R} = -\frac{G(\Delta M)}{R^2} = -\frac{G}{R^2}\left(\frac{4}{3}\pi R^3 \bar{\rho}\,\delta\right)$$

or $$\frac{\ddot{R}}{R} = -\frac{4}{3}\pi G \bar{\rho}\,\delta(t)$$

Thus, a mass excess $(\delta > 0)$



excess ($\delta > 0$)
causes collapse.



During collapse, mass must be conserved.

$$M = \frac{4}{3}\pi \bar{\rho}\,(1+\delta(t))\,R^3(t) = \text{const.}$$

$$\therefore \; R(t) = R_0\,(1+\delta(t))^{-1/3} \quad \text{where } R_0 = \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3} = \text{const.}$$

when $\delta \ll 1$, $R(t) = R_0\left(1 - \frac{1}{3}\delta(t)\right)$

$$\Rightarrow \quad \ddot{R} \simeq -\frac{1}{3}R_0\ddot{\delta} \simeq -\frac{1}{3}R\ddot{\delta}$$

$$\text{or } \frac{\ddot{R}}{R} \simeq -\frac{1}{3}\ddot{\delta} \quad \text{when } \delta \ll 1$$

Sub. into acc'n equ: $\ddot{\delta} = 4\pi G\bar{\rho}\,\delta$

Sol'n is $\delta(t) = A e^{t/t_{dyn}}$

Where $t_{dyn} = \dfrac{1}{(4\pi G\bar{\rho})^{1/2}}$ is the dynamical time for collapse

Note that it depends on $\bar{\rho}$ & not on R.
Runaway collapse can be halted by pressure, but that pressure gradient takes time to set up as pressure travels at the local speed of sound.

∴ The time it takes for the pressure gradient to build up in a radius R is $t_{pre} \sim \frac{R}{c_s}$

where $c_s = \sqrt{\frac{dP}{d\rho}}$ is the sound speed.

For hydrostatic equilibrium to be attained,

$$t_{pre} < t_{dyn}$$

Or, for a density perturbation to be stabilized by pressure against collapse it must be smaller than

$$\therefore \lambda_J = 2\pi c_s t_{dyn} = c_s \left(\frac{\pi}{G\bar{\rho}}\right)^{1/2}$$

λ_J = Jean's Length

Overdense regions larger than λ_J collapse
Overdense regions smaller than λ_J oscillate

When fluctuation occur in an expanding Univ., the expansion timescale becomes an important parameter e.g., in a spatially flat Univ., the characterstic time for expansion is the Hubble time

$H^{-1} = \left(\frac{3}{8\pi G\bar{\rho}}\right)^{1/2}$. This looks similar to the dynamical time for collapse $t_{dyn} = \left(\frac{1}{4\pi G\rho}\right)^{1/2}$

In fact, $H^{-1} = \left(\frac{3}{2}\right)^{1/2} t_{dyn} \simeq 1.22\, t_{dyn}$

The Jean's length in an expanding flat Univ. is then $\lambda_J = 2\pi c_s t_{dyn} = 2\pi \left(\frac{2}{3}\right)^{1/2} \frac{c_s}{H}$

e.g. the radiation comp. of the Univ:

$$P = \frac{\rho c^2}{3} \rightarrow c_s = \left(\frac{dP}{d\rho}\right)^{1/2} = \frac{c}{\sqrt{3}}$$

$$\therefore \lambda_J = \frac{2\pi\sqrt{2}}{3} \frac{c}{H} \simeq 3.0 \frac{c}{H}$$

ie, density fluctuations in the radiative component will be pressure supported if they are small than 3x the Hubble distance.

∴ A non-relativistic comp. must be responsible for collapsed structures much smaller than the Hubble distance.

However, prior to decoupling the baryons were coupled to the photons w/ $\rho_\gamma > \rho_b$ so λ_J was still $\simeq 3\frac{c}{H}$

After decoupling, the baryon fluid is seperate w/ $c_s = \left(\frac{kT}{mc^2}\right)^{1/2} c \simeq 1.5\times10^{-5} c$ @ $z_{dec.}$

dropping their λ_J by $\sim 3\times10^{-5}$ or $10^{-5}\frac{c}{H}$

$$M_J = \rho_{bary}\left(\frac{4}{3}\pi\lambda_J^3\right) \simeq 10^5 M_\odot$$

This dropped further as the Univ. expanded and baryons cooled.

→ After decoupling, the growth of density perturb. in the baryonic comp. took off.

Instability in an Expanding Univ.

Newtonian analysis. Consider a Univ. w/ pressureless matter w/ $\bar\rho(t) \propto a(t)^{-3}$. In a spherical region of radius R, add or remove a small amount of matter so in the sphere $\rho(t) = \bar\rho(t)(1+\delta(t))$ w/ $\delta \ll 1$

[also $R \ll \frac{c}{H}$ and $R > \lambda_J$]

The grav. acc'n at the surface of the sphere

$$\ddot R = -\frac{GM}{R^2} = -\frac{G}{R^2}\left(\frac{4}{3}\pi\rho R^3\right) = -\frac{4}{3}\pi G\rho R$$

$$- \frac{4\pi}{3} G(\bar{\rho}\delta) R$$

or $$\frac{\ddot{R}}{R} = -\frac{4}{3}\pi G\bar{\rho} - \frac{4}{3}\pi G\bar{\rho}\delta$$

Mass conservation inside the sphere

$$M = \frac{4}{3}\pi\bar{\rho}(t)\,(1+\delta(t))\,R^3(t) = \text{const.}$$

$$R(t) \propto \bar{\rho}(t)^{-1/3}\left[1+\delta(t)\right]^{-1/3}$$

or since $\bar{\rho} \propto a^{-3}$

$$R(t) \propto a(t)\left[1+\delta(t)\right]^{-1/3}$$

Sphere is always growing but δ can cause it to grow faster ($\delta < 0$) or slower ($\delta > 0$) than $a(t)$

Taking 2 time derivatives:

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} \qquad (\text{when } |\delta| \ll 1)$$

Combining w/ acc'n equation from above:

$$\frac{\ddot{a}}{a} - \frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} = -\frac{4}{3}\pi G\bar{\rho} - \frac{4}{3}\pi G\bar{\rho}\delta$$

[when $\delta = 0$, $\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G\bar{\rho}$ as before]

Subtracting normal acc'n equation from perturb. eqn gives the evol. of the perturbation:

$$-\frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} = -\frac{4\pi}{3}G\bar{\rho}\delta$$

or

$$\boxed{\ddot{\delta} + \underbrace{2H\dot{\delta}}_{\text{Hubble friction term}} = 4\pi G\bar{\rho}\delta}$$

since $H = \frac{\dot{a}}{a}$

GR calc'n gives same formula for pressureless matter.

In terms of Ω_m,

$$\boxed{\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0}$$