$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0$$

In a rad'n dom. Univ., $\Omega_m \ll 1$; $H = \frac{1}{2t}$

$$\ddot{\delta} + \frac{\dot{\delta}}{t} \simeq 0 \Rightarrow \delta \simeq B_1 + B_2 \ln t$$

Consider a flat, matter dominated Univ.

How do DM fluctuations grow?

$$\Omega_m = 1 \;;\; H = \frac{2}{3t} \longrightarrow \ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0$$

Assume a power-law sol'n Dt^n. Plugging in

$$n(n-1)Dt^{n-2} + \frac{4}{3t} n D t^{n-1} - \frac{2}{3t^2} D t^n = 0$$

$$\text{or} \quad n(n-1) + \frac{4}{3}n - \frac{2}{3} = 0$$

Quadratic eqn gives solutions $n = -1$; $\frac{2}{3}$

So, the general sol'n for the time evol. of density perturbations in a spatially-flat matter-only Univ. is $\delta(t) \simeq D_1 t^{2/3} + D_2 t^{-1}$

The decaying mode eventually becomes negligibly small $\rightarrow \delta \propto t^{2/3} \propto a(t) \propto \frac{1}{1+z}$ $\quad |\delta| \ll 1$

When an overdense region hits $\delta \sim 1$, can no longer use linear perturbation theory. Need computers. The overdense region breaks from the Hubble flow and collapses. After ~2 oscillations it obtains virial equilibrium. After decoupling baryons cool & fall in to existing collapsed DM halos to form galaxies, stars, etc.

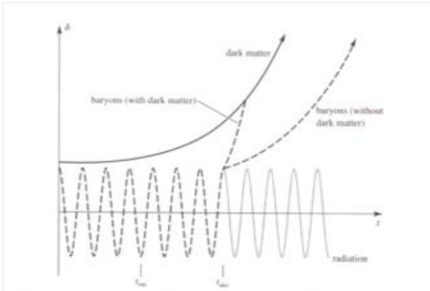


FIGURE 12.4 A highly schematic drawing of how density fluctuations in different components of the universe evolve with time.

The Power Spectrum of Density Fluctuations

The equation for the growth of δ doesn't depend on the shape or location of the perturbation. As seen from the CMB, the perturbations in the DM have a range of size & locations. Thus, there was an initial spectrum of perturbations that we can constrain by considering how it will evolve.

Consider early density fluctuations @ some time t_i, then at comoving location $\vec{r}$ there is a

$\delta(\vec{r})$ $[|\delta| \ll 1]$. The statistical properties of the field $\delta(\vec{r})$ are most important to cosmology, so decompose into Fourier components

$$\delta(\vec{r}) = \frac{V}{(2\pi)^3} \int \delta_{\vec{k}} \, e^{-i\vec{k}\cdot\vec{r}} \, d^3k$$

where V = large comoving box of volume V

and $$\delta_{\vec{k}} = \frac{1}{V} \int \delta(\vec{r}) \, e^{i\vec{k}\cdot\vec{r}} \, d^3r$$

$\vec{k}$ = comoving wavenumber

and $\lambda = \frac{2\pi}{k}$ is the wavelength of each Fourier component

Each Fourier component is a complex number

$$\delta_{\vec{k}} = |\delta_k| e^{i\phi_k}$$

When $|\delta_{\vec{k}}| \ll 1$, each component obeys

$$\ddot{\delta}_{\vec{k}} + 2H\dot{\delta}_{\vec{k}} - \frac{3}{2}\Omega_m H^2 \delta_{\vec{k}} = 0$$

as long as $a(t)\frac{2\pi}{k} > \lambda_J$ & small compared to $\frac{c}{H}$

The mean-square amplitude of the $\delta_{\vec{k}}$ defines

the Power spectrum $P(K) = \langle |\delta_{\vec{K}}|^2 \rangle$
where the average is taken over all orientations of $\vec{K}$ (fine if isotropic)

When the phases $\phi_{\vec{K}}$ of the diff. components are uncorrelated w/ each other, the $\delta(\vec{r})$ is called a Gaussian field. If a Gaussian field is homogenous & isotropic, then all of its statist. properties are contained in $P(K)$

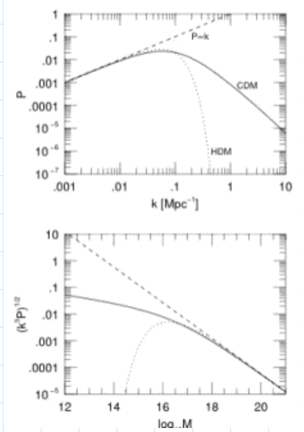
If $\delta(\vec{r})$ is a Gaussian field, then the value of δ @ a randomly selected point at t_i is $p(\delta) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\delta^2/2\sigma^2}$ where

$$\sigma = \frac{V}{(2\pi)^3} \int P(K)\, d^3K = \frac{V}{2\pi^2} \int_0^\infty P(K) K^2 \, dK$$

Most inflationary models predict density fluctuations will be an isotropic, homogeneous, Gaussian field. In addition, predict a scale-invariant $P(K) \propto K^n$ w/ $n=1$ being the favoured value (Harrison-Zeldovich spectrum)

The shape of the power spectrum will be modified b/w inflation & the time of radiation-matter equality. Depends on DM propertie (ie, hot or cold DM)

When the Univ. is rad'n dom., $\delta_{\vec{k}}$ donot grow quickly as long as $a(t)\frac{2\pi}{K} \ll \frac{c}{H(t)}$. But if, $a(t)\frac{2\pi}{K} \gg \frac{c}{H(t)}$ then $\delta_{\vec{k}}$ can increase if the DM particle is decoupled from rad'n



← at rad-matter equality

Mean mass ~~of~~ of a sphere $\langle M \rangle = \frac{4}{3}\pi L^3 \rho_m \propto K^{-3}$

but this is more interesting

$$\left\langle \left(\frac{M - \langle M \rangle}{\langle M \rangle} \right)^2 \right\rangle \propto K^3 P(K) \quad \text{where } K = \frac{2\pi}{L}$$

More on Large-Scale Structure

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G \bar{\rho}\, \delta \qquad |\delta| \ll 1$$

Note that this eqn. is independent of position or derivatives wrt position.

$\therefore$ Has solutions of the form

$$\delta(\vec{x}, t) = D(t)\, \tilde{\delta}(\vec{x})$$ where $\tilde{\delta}(\vec{x})$ is an arbitrary function of spatial coords.

$\therefore$ $D(t)$ satisfies

$$\ddot{D} + 2H\dot{D} - 4\pi G \bar{\rho}(t) D = 0$$

Only interested in growing sol'n, denoted by $D_+(t)$; normalized such that $D_+(t_0) = 1$

$\therefore \ \delta(\vec{x}, t) = D_+(t)\, \delta_0(\vec{x})$

$\rightarrow$ the spatial shape of the density fluctuations is frozen in comoving coords, only their amplitudes increase (if $|\delta| \ll 1$)

$\rightarrow$ $D_+(t)$ = growth factor

Can show that

$$D_+(a) \propto \frac{H}{H_0} \int_0^a \frac{da'}{\left[\Omega_{m,0}/a' + \Omega_{\Lambda,0}/a'^2 - (\Omega_{m,0} + \Omega_{\Lambda,0} - 1)\right]^{3/2}}$$

where prop. constant determined from $D_+(t_0) = 1$

- $\delta_0(\vec{x})$ = linearly extrapolated density fluctuation field

- e.g. $\Omega_{m,0} = 1$, $\Omega_{\Lambda,0} = 0$, $D_+(t) = \left(\frac{t}{t_0}\right)^{2/3}$