Consider spheres of radius $R = 8\,h^{-1}\,\mathrm{Mpc}$ in the local Universe. It is found that optically-selected galaxies have a fluctuation amplitude about 1,

$$\sigma_{8,g}^2 = \left\langle \left(\frac{\Delta n}{n} \right)^2 \right\rangle_8 \simeq 1$$

where the avg is over different spheres of radius $8\,h^{-1}\,\mathrm{Mpc}$. Then define $\sigma_8^2 = \langle \delta^2 \rangle_8$,

$$\text{so} \quad \sigma_8 = \frac{\sigma_{8,g}}{b} \simeq \frac{1}{b}$$

It is common to now use σ_8 as a parameter to normalize $P(k)$. Since $\sigma_8 \sim b \sim 1 - \delta$, fluctuations are becoming non-linear at $8\,h^{-1}\,\mathrm{Mpc}$ scales. On larger scales, linear perturbation theory is still applicable

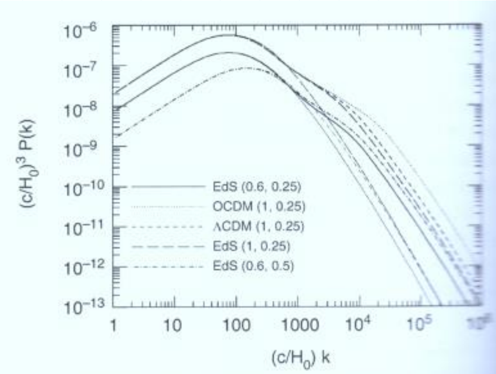
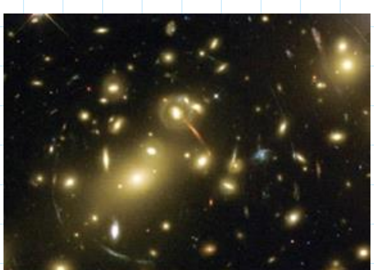


Fig. 7.6. The current power spectrum of density fluctuations for CDM models. The wave number k is given in units of H_0/c, and $(H_0/c)^3\,P(k)$ is dimensionless. The various curves have different cosmological parameters: EdS: $\Omega_m = 1$, $\Omega_\Lambda = 0$; OCDM: $\Omega_m = 0.3$, $\Omega_\Lambda = 0$; ΛCDM: $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$. The values in parentheses specify (σ_8, Γ), where σ_8 is the normalization of the power spectrum (which will be discussed below), and where Γ is the shape parameter. The thin curves correspond to the power spectrum $P_0(k)$ linearly extrapolated to the present day, and the bold curves take the non-linear evolution into account

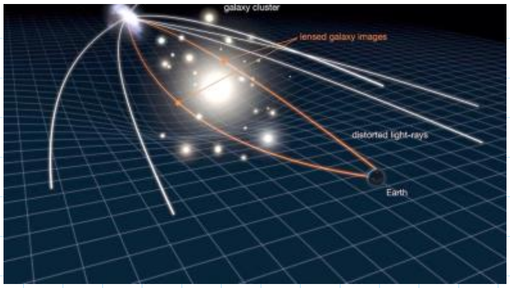
Gravitational Lensing

A prediction of GR. Distant sources have their light bent and magnified by a foreground object along the line of sight.



Gravitational lensing in galaxy cluster Abell 2218. NASA, A Fruchter and the ERO Team (STScI).

Consider light from a distant source passing very close to a lens and at that point space-time is



that point space-time is locally flat (ie, described by the Minkowski metric). except for a small perturbation due to the Newtonian potential of the lens Φ. Can construct an effective refractive index $n = 1 + \frac{2}{c^2}|\Phi|$ $\therefore$ $V_{(light)} = c - \frac{2}{c}|\Phi|$

From optics can get deflection angle of light

$$\vec{\hat{\alpha}} = \frac{2}{c^2}\int \vec{\nabla}_\perp \Phi \, dl$$

where $\vec{\nabla}_\perp \Phi$ is the gradient of the potential $\perp$ to the light path and integral is along the light path. Since the angles are so small, the integral is usually taken along the path of an undeflected ray with some impact parameter.

$\therefore$ angle depends on mass of lens. For a beam passing a point mass M at a distance ξ from the center of the lens

$$\hat{\alpha}(\xi) = \frac{4GM}{c^2 \xi}$$

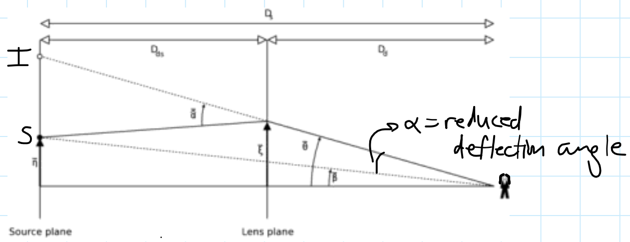
if $M = M_\odot$; $\xi = R_\odot$, $\hat{\alpha}_\odot \simeq 1''.74$

(cf. Eddington's measurement in 1919; w/ 30% accuracy)

For an extended mass dist'n in the lens and assuming the mass dist'n along the line-of-sight is very small compares to the distances involved, then the projected 2D mass dist'n Σ is the key to the deflection angle. The plane of this mass sheet is called the lens plane

$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi}')\,\Sigma(\vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2}\, d^2\xi'$$

But, can't measure $\hat{\alpha}$. Need to relate this to observables.



All distances are ang-diameter distances.

Then $\theta D_s = \beta D_s + \hat{\alpha} D_{ds}$ and $\alpha = \frac{D_{ds}}{D_s}\hat{\alpha}$

Obtain the lens equation $\boxed{\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})}$

In general this is a non-linear eqn & one can obtain multiple $\vec{\theta}_i$ for a true position $\vec{\beta}$, ie., get multiple images.

Point mass solution:

First, define the Einstein angle

$$\theta_E \equiv \sqrt{\frac{4GM}{c^2} \frac{D_{ds}}{D_s D_d}}$$

$\therefore\ \vec{\beta} = \vec{\theta} - \theta_E^2 \frac{\vec{\theta}}{\ }$; define $\vec{y} = \frac{\vec{\beta}}{\ }$ & $\vec{x} = \frac{\vec{\theta}}{\ }$

$\therefore\ \vec{\beta} = \vec{\theta} - \theta_E^2 \dfrac{\vec{\theta}}{|\vec{\theta}|^2}$; define $\vec{y} = \dfrac{\vec{\beta}}{\theta_E}$; $\vec{x} = \dfrac{\vec{\theta}}{\theta_E}$

then lens eq'n is $\vec{y} = \vec{x} - \dfrac{\vec{x}}{|\vec{x}|^2}$

Interesting case is $\vec{y} = 0$, then $|\vec{x}| = 1$ or $|\vec{\theta}| = \theta_E$ is a solution and image is a ring called an Einstein ring w/ avg. diameter $2\theta_E$

A more convienent way to write the scaled deflected is $\vec{\alpha}(\vec{\theta}) = \dfrac{1}{\pi}\int d^2\theta' \, \kappa(\vec{\theta}') \dfrac{\vec{\theta} - \vec{\theta}'}{|\vec{\theta} - \vec{\theta}'|^2}$ where

$\kappa_\theta = \dfrac{\Sigma(D_d \theta)}{\Sigma_{cr}}$ is the dimensionless surface density and the critical surface mass density

$\Sigma_{cr} = \dfrac{c^2 D_s}{4\pi G D_d D_{ds}}$ depends on the distances to the source & the lens

$\simeq 0.35 \left(\dfrac{D_d D_{ds}}{D_s / \text{Gpc}}\right)^{-1} \text{g cm}^{-2}$

A detailed analysis of the lens equations yields

A) if $\Sigma > \Sigma_{cr}$ in at least one point of the lens, then source positions $\vec{\beta}$ exist such that a source at $\vec{\beta}$ has multiple images. $\therefore$ If $\kappa \geq 1$ for certain regions of $\vec{\theta}$ lens is 'strong'

B) if $\kappa \ll 1$ at all points, lens is 'weak' and unable to produce multiple images