Return to $\dot{a}^2 = \frac{8\pi G \rho_0}{3} \frac{1}{a} - Kc^2$

- if $K < 0$, RHS > 0, and since $\dot{a} > 0$ now, $\dot{a} > 0$ always ∴ Univ. expands forever
  - i.e., $KE > PE$

- if $K = 0$, RHS > 0 and Univ. also expands forever, but in a way that $\dot{a} \to 0$ as $t \to \infty$ ($KE = PE$)

- if $K > 0$, RHS vanishes if $a = a_{max} = \frac{8\pi G \rho_0}{3Kc^2}$

  For this value of a, $\dot{a} = 0$ and the expansion will halt. Then contraction and universe will recollapse ($PE > KE$)

The Fluid Equation

Aside from gravity, the material that makes up the Univ also contributes to the overall evolution of $a(t)$.

1^{st} Law of Thermodynamics

$$dE + p\,dV = T\,dS$$

Apply this to an expanding volume V of unit comoving radius $\therefore$ radius $= a$

In general, energy in the volume is

$$E = mc^2 = \frac{4}{3}\pi a^3 \rho c^2$$

$$\frac{dE}{dt} = 4\pi a^2 \rho c^2 \dot{a} + \frac{4}{3}\pi a^3 \dot{\rho} c^2$$

and $$\frac{dV}{dt} = 4\pi a^2 \dot{a}$$

Assuming reversible adiabatic expansion $dS = 0$

$$\cancel{4\pi} a^2 \rho c^2 \dot{a} + \frac{\cancel{4}}{3}\cancel{\pi} a^3 \dot{\rho} c^2 + p\cancel{4\pi} a^2 \dot{a} = 0$$

$\div \frac{1}{3} a^3 c^2$

$$3\rho \frac{\dot{a}}{a} + \dot{\rho} + \frac{3p}{c^2}\frac{\dot{a}}{a} = 0$$

$$\boxed{\dot{\rho} + \frac{3\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0}$$

1^{st} term in brackets: dilution of density due to

1st term in brackets: dilution of density due to expansion

2nd term in brackets: loss of energy b/c the pressure has done work as the volume increased

No pressure forces in a homogeneous Univ.
→ no gradients!

→ pressure only drops energy density by the work done (energy goes into grav. p.e.)

To solve evol. eqns need to specify eqn. of state: $p = p(\rho)$ for a particular material

---

Acceleration Equation

F.E. $\dot{a}^2 = \frac{8\pi G \rho a^2}{3} - Kc^2$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G \rho}{3} - \frac{Kc^2}{a^2}$$

$\frac{d}{dt}$ both sides

$$\frac{2\dot{a}}{a}\,\frac{a\ddot{a} - \dot{a}^2}{a^2} = \frac{8\pi G}{3}\dot{\rho} + \frac{2Kc^2\dot{a}}{a^3}$$

sub in $\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$

$$\frac{\ }{a} \quad \frac{\ }{a^2} \quad \frac{\ }{3} \quad \frac{\ }{a^3}$$

sub in $\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)$

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G\left(\rho + \frac{p}{c^2}\right) + \frac{Kc^2}{a^2}$$

but $\frac{Kc^2}{a^2} = \frac{8\pi G\rho}{3} - \left(\frac{\dot{a}}{a}\right)^2$ from F.E.

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)}$$

Note that $p > 0$ increases the grav. force so further decelerates the expansion.

Equations of State

1) Matter, or, more precisely, non-relativistic matter, i.e. $\bar{v} \ll c$

In this case, $E \simeq mc^2 + \frac{1}{2}\frac{p_m^2}{m} \simeq mc^2$

where p_m = momentum and $\boxed{p = 0}$

(momentum is tiny compared to rest mass energy)

– baryonic matter is v. diffuse and cool

- only interactions are gravitational (also called 'dust')

2) Radiation, or relativistic matter (ie, neutrinos)

- momentum dominates over any rest mass energy

$$\boxed{P = \frac{\rho c^2}{3}}$$

Go back to fluid eq'n:

For matter, $\dot{\rho} + \frac{3\dot{a}}{a}\rho = 0 \rightarrow \frac{1}{a^3}\frac{d}{dt}(\rho a^3) = 0$

ie. $\rho a^3 = \text{const.}$ or $\rho = \frac{\rho_0 a_0}{a^3} = \frac{\rho_0}{a^3}$ w/ $a_0 = 1$

(cons. of mass eqn as before)

For radiation, $\dot{\rho} + \frac{4\dot{a}}{a}\rho = 0 \rightarrow \frac{1}{a^4}\frac{d}{dt}(\rho a^4) = 0$

ie $\rho a^4 = \text{const}$ or $\rho = \frac{\rho_0}{a^4}$ w/ $a_0 = 1$

Why the extra power in a?

- extra loss in energy due to pdV work
- or, since $E = \frac{hc}{\lambda}$; λ is stretched by

$a \uparrow$, $E \downarrow$ for each photon