Monday, August 24, 2015 1:06 PM

Time Evolutions : Want $a(t)$

Start w/ F.E. : $\dot{a}^2 = \frac{8\pi G\rho a^2}{3} - Kc^2$

choose matter domination: $\rho = \frac{\rho_0}{a^3}$ (set $a_0 = 1$)

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3a} - Kc^2 = A(a(t))$$

$$\frac{da}{dt} = \sqrt{A(a(t))}$$

$$dt = \frac{da}{\sqrt{A(a(t))}}$$

$$t - t_1 = \int_{a_1}^{a} \frac{da}{\left(\frac{8\pi G\rho_0}{3a} - Kc^2\right)^{1/2}}$$

let $a_m = \frac{8\pi G\rho_0}{3c^2}$ also let $a = 0$ at $t = 0$

$$\therefore \int_0^a \frac{da}{c\left(\frac{a_m}{a} - K\right)^{1/2}} = \int_0^a \frac{da}{c\left(\frac{a_m}{a}\right)^{1/2}\left(1 - \frac{Ka}{a_m}\right)^{1/2}}$$

Consider $K = 0$

$$t = \int \frac{a^{1/2}\,da}{c\,a_m^{1/2}} \rightarrow t = \frac{2}{3}\frac{a^{3/2}}{c\,a_m^{1/2}}$$

$$\left(3 \quad a^{1/2}\right) t^{2/3}$$

or $\underline{a = \left(\frac{3}{2} c a_m^{1/2}\right) t^{2/3}}$

if $K=+1$, let $\frac{a}{a_m} = \sin^2\theta$

$$da = 2a_m \sin\theta\cos\theta\, d\theta \Rightarrow \theta = \arcsin\left(\left(\frac{a}{a_m}\right)^{1/2}\right)$$

$$\therefore t = \int_0^{\sin^{-1}[(a/a_m)^{1/2}]} \frac{2a_m \sin\theta\cos\theta\, d\theta}{\frac{c}{\sin\theta}(1-\sin^2\theta)^{1/2}} = \int_0^{\sin^{-1}[(a/a_m)^{1/2}]} \frac{2a_m \sin^2\theta\, d\theta}{c}$$

$$\int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta = \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta$$

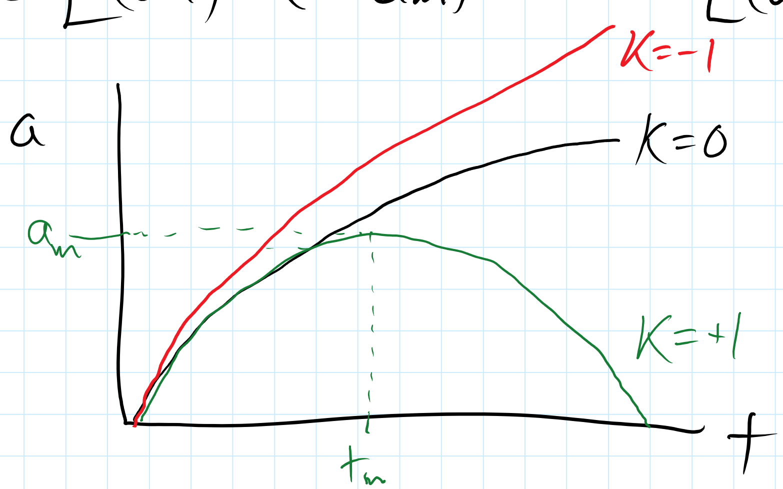
$$\therefore t = \frac{a_m}{c}\left[\sin^{-1}\left[\left(\frac{a}{a_m}\right)^{1/2}\right] - \left(\frac{a}{a_m}\right)^{1/2}\left(1-\frac{a}{a_m}\right)^{1/2}\right]$$

Consider $K=-1$, let $\frac{a}{a_m} = \sinh^2\theta$

use $\cosh^2\theta - \sinh^2\theta = 1$

end up w/ $\int \sinh^2\theta\, d\theta = \frac{1}{2}\sinh\theta\cosh\theta - \frac{\theta}{2}$

$$t = \frac{a_m}{c}\left[\left(\frac{a}{a_m}\right)^{1/2}\left(1+\frac{a}{a_m}\right)^{1/2} - \sinh^{-1}\left[\left(\frac{a}{a_m}\right)^{1/2}\right]\right]$$



Time Evolution, rad'n dominate

$$\dot{a}^2 = \frac{8}{3}\pi G \rho a^2 - Kc^2 \quad ; \quad \rho = \frac{\rho_0}{a^4}$$

$$\dot{a}^2 = \frac{8}{3}\pi G \frac{\rho_0}{a^2} - Kc^2$$

$$K=0: \quad \frac{da}{dt} = \sqrt{\frac{8}{3}\pi G \rho_0}\,\frac{1}{a}$$

$$\int_0^a a\,da = \sqrt{\tfrac{8}{3}\pi G \rho_0}\int_0^t dt$$

$$\frac{1}{2}a^2 = \sqrt{\tfrac{8}{3}\pi G \rho_0}\, t \;\longrightarrow\; a = \sqrt{2}\left(\tfrac{8}{3}\pi G \rho_0\right)^{1/4} t^{1/2}$$

Univ. expands more slowly if rad'n dom. then if matter dom.

– extra decelleration from pressure

Mixtures of Rad'n & Matter

Then $\rho = \rho_{rad'n} + \rho_{matter}$ (hard, in general)

Consider the simple situation where one or the other densities is larger ($K=0$)

So, if rad'n dominates the expansion,

$$a(t) \propto t^{1/2}, \quad \rho_{rad} \propto \frac{1}{a^4} \propto \frac{1}{t^2}, \quad \rho_{matt} \propto \frac{1}{a^3} \propto \frac{1}{t^{3/2}}$$

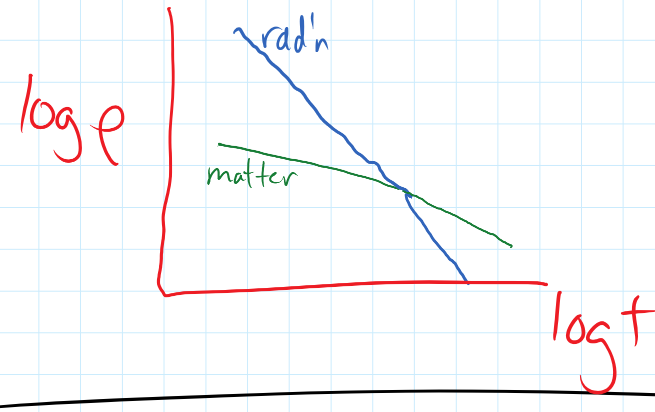
Radiation density falls off faster than matter

∴ radiation domination cannot last forever (unstable) → matter will eventually dominate

If matter dominates:

$a(t) \propto t^{2/3}$; $\rho_{rad} \propto \frac{1}{a^4} \propto \frac{1}{t^{8/3}}$; $\rho_{matt} \propto \frac{1}{a^3} \propto \frac{1}{t^2}$

Matter will always dominate (stable)



Recall that a isotropic & homogeneous expanding Univ. gives Hubble's Law: $V = H_0 d$ where $H_0 = \frac{\dot{a}_0}{a_0}$

This is true at any time: $H(t) = \frac{\dot{a}}{a}$

where $H(t)$ = Hubble parameter.

Then the F.E. can be written as

$$H^2 = \frac{8\pi G\rho}{3} - \frac{Kc^2}{a^2}$$

(Aside: Often the Hubble constant H_0 is parametrized as $H_0 = 100\,h$ Km s^{-1} Mpc^{-1} where 0.732 ± 0.017 is an observational measurement) $h \simeq 0.7$

Useful to allow scalings to diff. values of H_0.
(ie, to compare to earlier results)

The smaller h is the slower the Univ. is
expanding. h is often seen in numbers in Cosmology