Start again $H^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2}$

rewrite this $\frac{a^2H^2}{c^2} = \frac{8\pi G a^2}{3c^2}\rho - K$

or $K = \frac{8\pi G a^2}{3c^2}\rho - \frac{a^2H^2}{c^2}$

For a given value of H, there is a special density required to make $K=0$. This is the critical density:

$$0 = \frac{8\pi G a^2 \rho_c}{3c^2} - \frac{a^2H^2}{c^2}$$

$$\therefore \rho_c = \frac{3H^2}{8\pi G}$$ NB: this is a function of time

Currently, $\rho_c(t_0) = 1.88\,h^2 \times 10^{-26}\ Kg\,m^{-3}$

(Compare to H_2O: $\rho = 10^3\ Kg/m^3$)

Re-write in astro units (check!)

$$\rho_c(t_0) = 2.78\,h^2 \times 10^{11}\ M_\odot\,Mpc^{-3}$$

$10^{11-12} M_\odot \simeq$ mass ~~of~~ a typical galaxy

Mpc $\simeq$ typ. galaxy seperation

so $\rho \sim \rho_c$ probably...

so $\rho \sim \rho_c$ probably...

ρ_c sets a natural scale for densities in the Univ. $\therefore$ Define the density parameter

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)}$$

The present value of Ω is Ω_0

Now $H^2 = \frac{8\pi G}{3} \rho_c \Omega - \frac{Kc^2}{a^2} = H^2 \Omega - \frac{Kc^2}{a^2}$

or $$\boxed{[\Omega - 1] = \frac{Kc^2}{a^2 H^2}}$$

if $\Omega = 1$, $K = 0$ at all times, 'critical univ.'

if $\Omega < 1$, $K < 0$ at all times

if $\Omega > 1$, $K > 0$ at all times

Measuring Ω now (ie, Ω_0) tells us about the fate of the Universe.

If $K \neq 0$, Ω can take any value.

Can have different Ωs, $\Omega_m = \frac{\rho_m}{\rho_c}$, $\Omega_{rad} = \frac{\rho_{rad}}{\rho_c}$

$$\Omega_{dm} = \frac{\rho_{dm}}{\rho_c}, \quad \Omega_r = \frac{\rho_r}{\rho_c} \quad ; \quad \Omega = \sum_i \Omega_i$$

often $\Omega_m = \Omega_{bary} + \Omega_{dm}$

The Deceleration Parameter, q_0

Taylor expansion of a around t_0:

$$a(t) = a(t_0) + \dot{a}(t_0)[t - t_0] + \frac{1}{2}\ddot{a}(t_0)[t - t_0]^2 + \cdots$$

$\div$ both sides by $a(t_0)$

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t - t_0) + \frac{\frac{1}{2}\ddot{a}(t_0)}{a(t_0)}(t - t_0)^2 + \cdots$$

$$= 1 + H_0(t - t_0) + \frac{1}{2}\frac{\ddot{a}(t_0)H_0^2}{a(t_0)H_0^2}(t - t_0)^2 + \cdots$$

Define $q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)}\frac{1}{H_0^2} = -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}^2(t_0)}$

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2 (t - t_0)^2 + \cdots$$

The larger q_0 is, the more rapid the deceleration.

Recall the acc'n eqn: $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$

For matter, $p = 0$, $\frac{\ddot{a}}{a} = -\frac{4\pi G \rho}{3}$

So, $\quad \dfrac{4\pi G \rho_0}{3} \dfrac{1}{H_0^2} \quad$ but $\rho_{c,0} = \dfrac{3H_0^2}{8\pi G}$

$$\frac{3}{8\pi G \rho_{c,0}} = \frac{1}{H_0^2}$$

$$\therefore \quad \frac{4\pi G \rho_0}{3} \cdot \frac{3}{8\pi G \rho_{c,0}} = \frac{1}{2} \frac{\rho_0}{\rho_{c,0}} = \frac{1}{2} \Omega_0 \quad (\text{really, } \Omega_{m,0})$$

If we know the prop. of the matter that dominates the Univ, q_0 is not independent of H_0 & Ω_0.

If we don't know, q_0 can tell us about acc'n or deceleration indep. of form of energy density.

Age of Univ.

Consider matter dominated $K=0$ Univ. $(\Omega_0 = 1)$

$$t = \frac{2}{3} \frac{a_m}{c} \left(\frac{a}{a_m} \right)^{3/2}$$

where $a_m = \dfrac{8\pi G \rho_0}{3c^2}$, So, $t_0 = \dfrac{2}{3} \dfrac{a_m}{c} \left(\dfrac{1}{a_m} \right)^{3/2}$

since $a_0 = 1$ by def'n.

$$\therefore \quad t_0 = \frac{2}{3} \frac{a_m^{-1/2}}{c} \quad \text{but} \quad a_m = \frac{8\pi G \rho_0 H_0^2}{3c^2 H_0^2}$$

$$= \Omega_0 \frac{H_0^2}{c^2} = \frac{H_0^2}{c^2}$$

$$= \Omega_0 \frac{H_0^2}{c^2} = \frac{H_0^2}{c^2}$$

$$\boxed{t_0 = \frac{2}{3}\frac{c}{H_0}\frac{1}{c} = \frac{2}{3}\frac{1}{H_0}}$$

At any time,

$$t = \frac{2}{3}\frac{1}{H}$$

Another example, an empty Univ. $\rho = 0$ or $\Omega = 0$
(Milne Univ.)

$$\dot{a}^2 = \cancel{\frac{8\pi G\rho a}{3}}^{\,0} - Kc^2$$

take $K = -1$, $\therefore \dot{a}^2 = c^2$ or $\dot{a} = c = \frac{da}{dt}$

$$\int_0^t dt = \int_0^a \frac{da}{c} \rightarrow t_0 = \frac{a_0}{c}$$

Recall $\frac{Kc^2}{a_0^2} = H_0^2(\Omega_0 - 1)$

$\Omega_0 = 0$, $K = -1$, $\frac{c^2}{a_0^2} = H_0^2$ or $\frac{a_0}{c} = \frac{1}{H_0}$

$$\therefore \quad t_0 = \frac{1}{H_0}$$

Rad'n: $K = 0$

$$a = \sqrt{2}\left(\frac{8}{3}\pi G\rho_0\right)^{1/4} t^{1/2}$$

but from F.E., $H_0^2 = \frac{8\pi G\rho_0}{3}$

$$\therefore \; a = \sqrt{2H_0 t}$$

at $t = t_0$, $a_0 = 1$, $\rightarrow \boxed{t_0 = \frac{1}{2H_0}}$