What is the meaning of K?

Compute the ratio of circumference to radius of a circle w/ comoving radius r_1.

proper radius $\ell = a(t)\int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}}$

circumference $dC = a(t)\, r\, d\phi \rightarrow C = 2\pi a(t) r_1$

so $\frac{C}{\ell} = \frac{2\pi a r_1}{a f(r_1)} = \frac{2\pi r_1}{f(r_1)}$

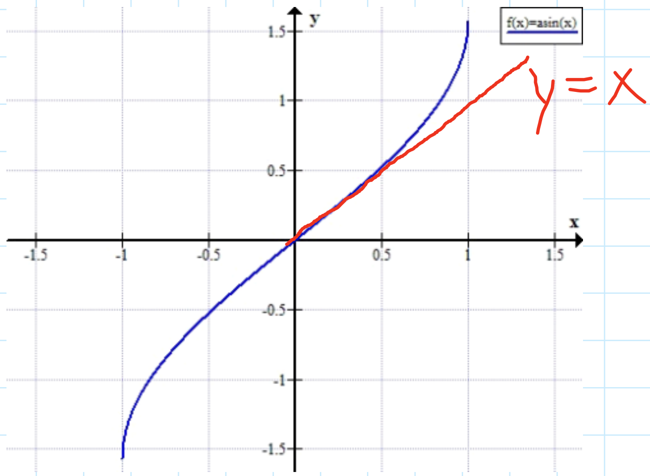
if $K=0$, $\frac{C}{\ell} = 2\pi$ (flat, Euclidea space)

$K=1$, $\frac{C}{\ell} = \frac{2\pi r_1}{\sin^{-1} r_1}$; $\sin^{-1} x > x$

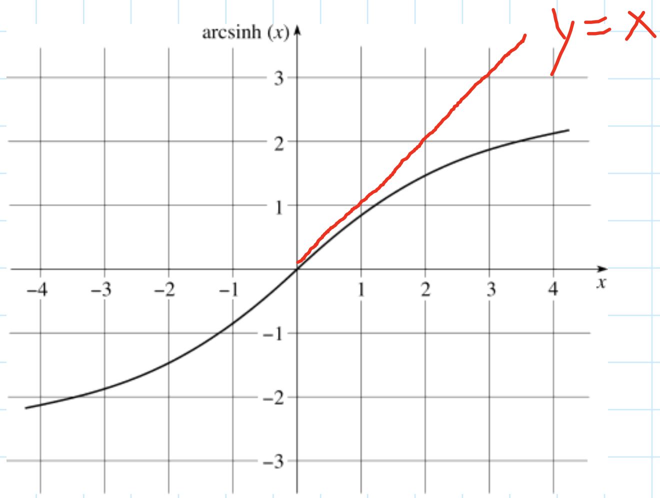
$\rightarrow \frac{C}{\ell} < 2\pi$

So on a curved surface

- positive curvature
- 2D analogue is the surface of a sphere



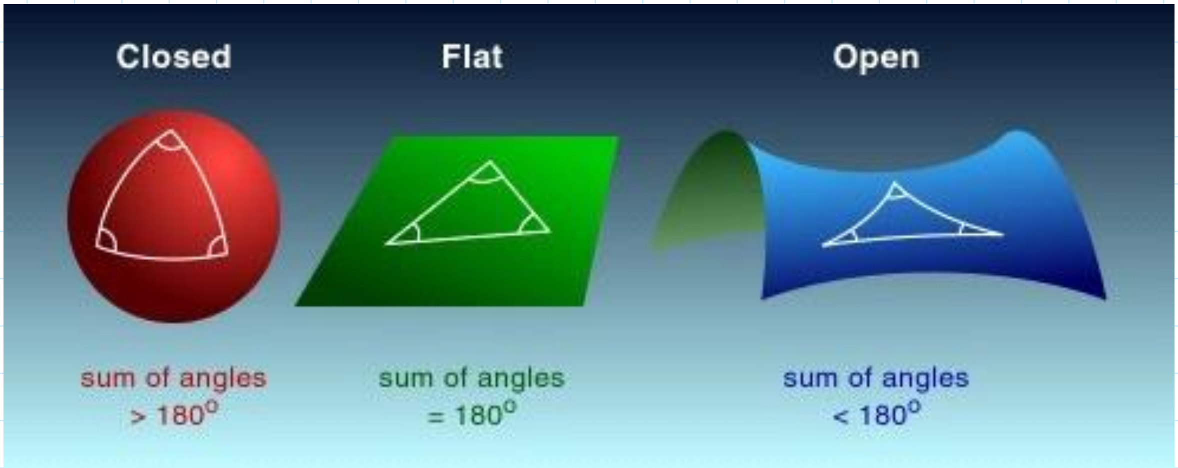
if $K=-1$, $\frac{C}{\ell} = \frac{2\pi r_1}{\sinh^{-1} r_1}$



here, $\frac{r_1}{\sinh^{-1} r_1} > 1$

so $\frac{C}{\ell} > 2\pi$

- curved space, 'negative curvature'
- 2D: saddle-like surface



$K=1$
+ve curv.
space is finite but unbounded

$K=0$
$\Omega_0 = 1$
space is flat & infinite

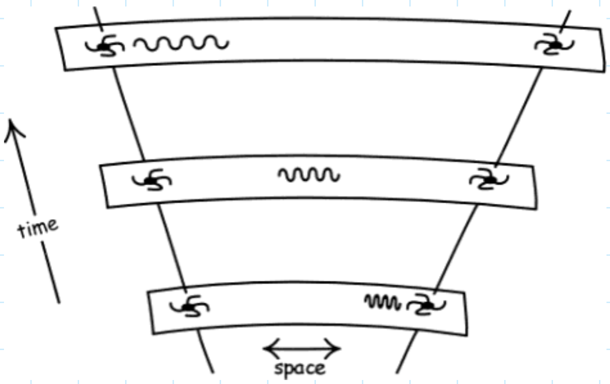
$K=-1$
-ve curvature
space is hyperbolic and infinite

Redshift

Redshift

assume 2 crests observed at

t_0 & $t_0 + \Delta t_0$



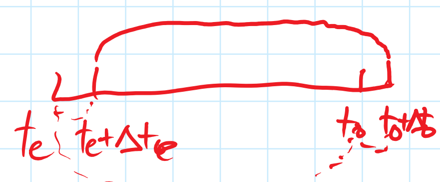
2 crests emitted at t_e & $t_e + \Delta t_e$

To describe the propogation of light set $ds=0$ in RW metric: $c\,dt = \dfrac{a(t)\,dr}{\sqrt{1-Kr^2}}$

1^{st} crest: $\displaystyle\int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-Kr^2}} = f(r_1)$

2^{nd} crest: $\displaystyle\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{c\,dt}{a(t)} = f(r_1)$

$$\rightarrow \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = \int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{c\,dt}{a(t)}$$



but $\displaystyle\int^{t_0+\Delta t_0} = \cancel{\int^{t_0}} - \int^{t_e+\Delta t_e} + \int^{t_0+\Delta t_0} = \cancel{\int^{t_0}}$

but $\int_{t_e+\Delta t_e} = \cancel{\int_{t_e}} - \int_{t_e} + \int_{t_0} = \cancel{\int_{t_e}}$

$$\therefore \int_{t_e}^{t_e+\Delta t_e} \frac{c\,dt}{a(t)} = \int_{t_0}^{t_0+\Delta t_0} \frac{c\,dt}{a(t)}$$

a can be considered to be constant over such small time intervals

$$\frac{c\,\Delta t_e}{a(t_e)} = \frac{c\,\Delta t_0}{a(t_0)}$$

or $$\frac{c\,\Delta t_e}{c\,\Delta t_0} = \frac{a(t_e)}{a(t_0)}$$

$$\Delta t_e = \frac{1}{\nu_e} \;;\; \Delta t_0 = \frac{1}{\nu_0}$$

and since $\lambda = \frac{c}{\nu}$, $\lambda_e = c\,\Delta t_e$; $\lambda_0 = c\,\Delta t_0$

$$\therefore \frac{\lambda_e}{\lambda_0} = \frac{a(t_e)}{a(t_0)}$$ → redshift in expanding Univ.

define $z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\lambda_0}{\lambda_e} - 1$ redshift

$$\therefore z = a(t_0) - 1$$

$$\therefore \quad z = \frac{a(t_0)}{a(t_e)} - 1$$

or $$\boxed{1+z = \frac{a(t_0)}{a(t_e)}}$$

From Einstein Field Eqns & R-W metric

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} + \frac{8\pi G P}{c^2} = -\frac{Kc^2}{a^2} + \Lambda \qquad (1)$$

and $$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\rho}{3} = -\frac{Kc^2}{a^2} + \frac{\Lambda}{3} \qquad (2)$$

(2) is our F.E. w/ cosmo. constant

Subtract (2) from (1)

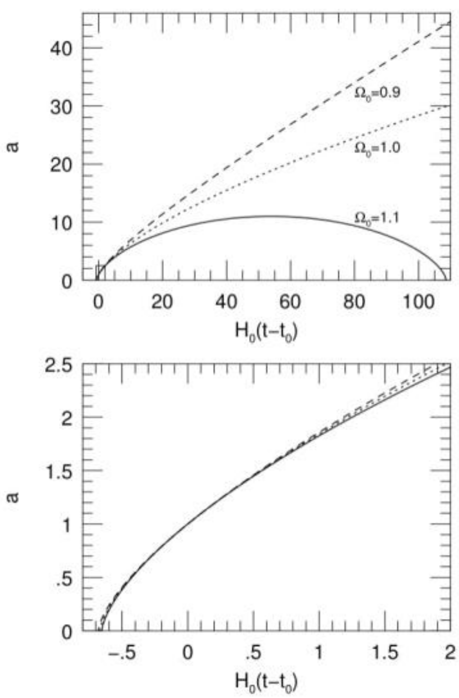
$$\frac{\ddot{a}}{a} + \frac{4\pi G}{3}\left[\rho + \frac{3P}{c^2}\right] - \frac{\Lambda}{3} = 0$$

Depending on sign of Λ, cosmo. const. can help w/ deceleration or provide an acc'n

[$\Lambda = 0$ gives standard eqn's]

Recall: $$(\Omega - 1) = \frac{Kc^2}{a^2H^2} \qquad \Lambda = 0$$

if $\Omega = 1$, $K=0$ Univ is



if $\Omega = 1$, $K=0$, Univ. is flat

$\Omega > 1$, $K > 0$, Univ. is closed

$\Omega < 1$, $K < 0$, Univ. is open

Λ Only Univ. (de Sitter Univ.)

F.E. $$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - \frac{Kc^2}{a^2}$$

assume $K=0$; Λ dominates the energy density

$$\dot{a}^2 = \frac{\Lambda a^2}{3}$$

$$\frac{da}{dt} = \sqrt{\frac{\Lambda}{3}}\, a$$

$$\int_{a_0}^{a} da = \sqrt{\frac{\Lambda}{3}} \int_{t_0}^{t} dt$$

$$\frac{a}{a_0} = e^{\sqrt{\frac{\Lambda}{3}}(t-t_0)}$$

or $a = e^{\sqrt{\frac{\Lambda}{3}}(t-t_0)}$

w/ $a_0 = 1$

$$\left.\begin{array}{l} \quad a_0 \qquad\qquad\qquad t_0 \\ \ln \frac{a}{a_0} = \sqrt{\frac{\Lambda}{3}}\,(t - t_0) \end{array}\right\}$$

w/ $a_0 = 1$

Exp. expansion!

Can also define a density parameter for Λ:

$$\Omega_\Lambda = \frac{\Lambda}{3H^2}$$

Although Λ is a const., Ω_Λ is not b/c $H(t)$

F.E. $$H^2 = \frac{8\pi G}{3}\rho_c \Omega_m + \frac{\Lambda}{3} - \frac{Kc^2}{a^2}$$

$$= H^2\Omega_m - \frac{Kc^2}{a^2} + \frac{\Lambda}{3H^2}H^2 = H^2\Omega_m - \frac{Kc^2}{a^2} + \Omega_\Lambda H^2$$

$$\therefore \left(\Omega_m + \Omega_\Lambda - 1\right) = \frac{Kc^2}{a^2H^2}$$

for $K=0$, $\Omega_m + \Omega_\Lambda = 1$