Phys. 4247 - Midterm #1 Solutions

1. Already know that $\rho_m = \rho_{m,0}\left(\frac{a_0}{a}\right)^3$ from the fluid equation.

Need to find analogous relation for quintessence. Use fluid eq'n.

$$\dot{\rho} + \frac{3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0 \qquad \dot{\rho} + \frac{3\dot{a}}{a}\frac{1}{2}\rho = 0$$

$$\dot{\rho} + \frac{3\dot{a}}{a}\left(\rho - \frac{1}{2}\rho\frac{c^2}{c^2}\right) = 0 \qquad \dot{\rho} + \frac{3}{2}\rho\frac{\dot{a}}{a} = 0$$

$$\therefore \dot{\rho} = -\frac{3}{2}\rho\frac{\dot{a}}{a} \rightarrow \frac{\dot{\rho}}{\rho} = -\frac{3}{2}\frac{\dot{a}}{a} \rightarrow \int_{\rho_{Q,0}}^{\rho_Q}\frac{d\rho}{\rho} = -\frac{3}{2}\int_{a_0}^{a}\frac{da}{a}$$

$$\therefore \ln\left(\frac{\rho_Q}{\rho_{Q,0}}\right) = -\frac{3}{2}\ln\left(\frac{a}{a_0}\right) \rightarrow \frac{\rho_Q}{\rho_{Q,0}} = \left(\frac{a}{a_0}\right)^{-3/2} \rightarrow \rho_Q = \rho_{Q,0}\left(\frac{a_0}{a}\right)^{3/2}$$

OK. When $a = a_{mQ}$, $\rho_Q = \rho_m$, so

$$\rho_{m,0}\left(\frac{a_0}{a_{mQ}}\right)^3 = \rho_{Q,0}\left(\frac{a_0}{a_{mQ}}\right)^{3/2}$$

$\div$ both sides by $\rho_{c,0}$, the critical density at the current time

$$\frac{\rho_{m,0}}{\rho_{c,0}}\left(\frac{a_0}{a_{mQ}}\right)^3 = \frac{\rho_{Q,0}}{\rho_{c,0}}\left(\frac{a_0}{a_{mQ}}\right)^{3/2} \rightarrow \Omega_{m,0}\left(\frac{a_0}{a_{mQ}}\right)^3 = \Omega_{Q,0}\left(\frac{a_0}{a_{mQ}}\right)^{3/2}$$

or $$\frac{\Omega_{m,0}}{(1-\Omega_{m,0})}\left(\frac{a_0}{a_{mQ}}\right)^3 = \left(\frac{a_0}{a_{mQ}}\right)^{3/2} \rightarrow \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3}\left(\frac{a_0}{a_{mQ}}\right) = \frac{a_0^{1/2}}{a_{mQ}^{1/2}}$$

$$a_{mQ}^{1/2} = a_0^{1/2}\left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3} \rightarrow a_{mQ} = a_0\left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{2/3}$$

2. Need acceleration equation w/ Λ.

F.E. w/ Λ: $\dot{a}^2 = \frac{8\pi G\rho a^2}{3} - Kc^2 + \frac{\Lambda a^2}{3}$

Differentiate w.r.t. time:

$$2\dot{a}\ddot{a} = \frac{8\pi G\rho 2a\dot{a}}{3} + \frac{8\pi G\dot{\rho}a^2}{3} + \frac{2a\dot{a}\Lambda}{3}$$

$\div$ by $2\dot{a}a$: $\frac{\ddot{a}}{a} = \frac{8\pi G\rho}{3} + \frac{8\pi G\dot{\rho}a}{3(2\dot{a})} + \frac{\Lambda}{3}$

Fluid eq'n: $\dot{\rho} + \frac{3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0 \rightarrow \dot{\rho} = -\frac{3\dot{a}\rho}{a}$ in pressureless Univ.

$$\therefore \frac{\ddot{a}}{a} = \frac{8\pi G\rho}{3} + \frac{8\pi G}{3}\left(\frac{-3\cancel{\dot{a}}\rho}{\cancel{a}}\right)\frac{\cancel{a}}{2\cancel{\dot{a}}} + \frac{\Lambda}{3}$$

$$= \frac{8\pi G\rho}{3} - \frac{12\pi G\rho}{3} + \frac{\Lambda}{3} = -\frac{4\pi G\rho}{3} + \frac{\Lambda}{3}$$

at current time: $\frac{\ddot{a}_0}{a_0} = -\frac{4\pi G\rho_0}{3}\frac{H_0^2}{H_0^2} + \frac{\Lambda_0}{3}\frac{H_0^2}{H_0^2} = -\frac{\rho_0}{2\rho_{c,0}}H_0^2 + \Omega_{\Lambda,0}H_0^2$

$$= -\frac{\Omega_{m,0}}{2}H_0^2 + \Omega_{\Lambda,0}H_0^2$$

We used $\rho_{c,0} = \frac{3H_0^2}{8\pi G}$; $\Omega_{\Lambda,0} = \frac{\Lambda_0}{3H_0^2}$

$$q_0 = -\frac{\ddot{a}_0}{a_0}\frac{1}{H_0^2} = \frac{\Omega_{m,0}}{2} - \Omega_{\Lambda,0}$$

3. For light travelling in an expanding Universe:

$$\int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = \int_{r_e}^{r_0} \frac{dr}{\sqrt{1-Kr^2}}$$

Here, $K=0$ so RHS is just $(r_0 - r_e) = \Delta r$, the cooridinate distance traversed by the light.

For the LHS, change variables so we are integrating over a instead of t: $\dot{a} = \frac{da}{dt} \rightarrow dt = \frac{da}{\dot{a}}$. $\therefore$ LHS $= \int_{a_e}^{a_0} \frac{c\,da}{a\dot{a}}$

$\dot{a}$ is found from the F.E. Recall $K=0$; $\Lambda = 0$ in this case

$$\therefore \dot{a}^2 = \frac{8\pi G \rho a^2}{3} = \frac{8\pi G \rho_0}{3a} \qquad \left(\text{Matter dominated so } \rho = \rho_0 \left(\frac{a_0}{a}\right)^3 \text{ and } a_0 = 1\right)$$

$$\therefore \text{LHS} = c\int_{a_e}^{a_0} \frac{da}{a\sqrt{\frac{8\pi G\rho_0}{3a}}} = c\sqrt{\frac{3}{8\pi G\rho_0}} \int_{a_e}^{a_0} \frac{da}{a^{1/2}} = c\sqrt{\frac{3H_0^2}{8\pi G\rho_0 H_0^2}} \left[2a^{1/2}\right]_{a_e}^{a_0}$$

$$= 2c\sqrt{\frac{\rho_{c,0}}{\rho_0 H_0^2}} \left(a_0^{1/2} - a_e^{1/2}\right) = \frac{2c}{\Omega_{m,0}^{1/2} H_0} a_0^{1/2} \left(1 - \left(\frac{a_e}{a_0}\right)^{1/2}\right)$$

But, in a flat, matter-dominated Universe w/ $\Lambda = 0$, $\Omega_{m,0} = 1$

$\left(\text{recall } (\Omega_{m,0} - 1) = \frac{K}{a_0^2 H_0^2}\right)$, and $1+z = \frac{a_0}{a_e} \rightarrow \frac{a_e}{a_0} = \frac{1}{1+z}$

$$\therefore \underline{\Delta r = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)} \qquad (\text{again, set } a_0 = 1)$$