Phys. 4247 - Assignment #1 Solutions

1) a) In an infinite distribution of galaxies, every line-of-sight should end at a galaxy; this would imply a surface brightness for the sky = surface brightness of a typical galaxy. From class, the flux/solid angle from a shell of galaxies w/ avg. number density n_* is:

$$dJ(r) = \frac{L}{4\pi r^2} n_* r^2 dr = \frac{n_* L}{4\pi} dr$$

The surface brightness of a typical galaxy is its flux/solid angle

ie, $\frac{L}{4\pi r^2} \Big/ \left(\pi R_g^2 / r^2\right) = \frac{L}{4\pi(\pi R_g^2)}$

Now, integrate $dJ(r)$ until the integrated surface brightness is equal to $\frac{L}{4\pi(\pi R_g^2)}$

$$J = \int_{r=0}^{r=r_{max}} dJ = \frac{nL}{4\pi}\int_{r=0}^{r_{max}} dr = \frac{n\cancel{L} r_{max}}{\cancel{4\pi}} = \frac{\cancel{L}}{\cancel{4\pi}(\pi R_g^2)}$$

$$\therefore r_{max} = \frac{1}{n_* \pi R_g^2}$$

[This can also be derived using the idea of a mean-free path of a radial ray]

Plugging in the numbers ($n_* = 1\ Mpc^{-3}$ and $R_g = 2000 pc$) you get $r_{max} = 8 \times 10^4\ Mpc$

(b) Start with the Friedmann Equation:

$$\dot{a}^2 = \frac{8\pi G\rho a^2}{3} - \cancel{Kc^2}^{\,0}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3}$$

Evaluate at the current time and set $H = \frac{\dot{a}}{a}$

$$H_0^2 = \frac{8\pi G\rho_0}{3}$$

$$\therefore \rho_0 = \frac{3H_0^2}{8\pi G}$$

$$H_0 = 70 \frac{Km}{s\,Mpc}\left(\frac{1000\ m}{1\ Km}\right)\left(\frac{1\ Mpc}{3.0826\times10^{22}\ m}\right) = 2.27\times10^{-18}\ s^{-1}$$

$$\therefore \rho_0 = \frac{3(2.27\times10^{-18}\ s^{-1})^2}{8\pi\,(6.67\times10^{-11}\ m^3\,Kg^{-1}\,s^{-2})} = 9.2\times10^{-27}\ Kg\ m^{-3}$$

$$\therefore n_{bb} = \frac{\rho_0}{m_{bb}} = 6.4\times10^{-26}\ \frac{1}{m^3}\left(\frac{3.086\times10^{22}\ m}{1\ Mpc}\right)^3 = \underline{1.9\times10^{42}\ Mpc^{-3}}$$

From part (a), the average distance that one could observe before seeing a baseball is

$$r_{max} = \frac{1}{n_{bb}\pi r_{bb}^2} = \frac{1}{(1.9\times10^{42}\ Mpc^{-3})\,\pi\,(0.0369m)^2\left(\frac{1\ Mpc}{3.086\times10^{22}\ m}\right)^2}$$

$$= \underline{1.2\times10^{5}\ Mpc}$$

2) Consider a ray propogating from a galaxy at redshift z to us. It starts w/ energy E_{em} and loses energy as $\frac{dE}{dr} = -KE$

$\therefore$ Its observed energy is found from $\int_{E_{em}}^{E_{obs}} \frac{dE}{E} = -K\int_0^r dr$

$$\ln E\Big]_{E_{em}}^{E_{obs}} = -Kr \rightarrow \ln\left(\frac{E_{obs}}{E_{em}}\right) = -Kr \rightarrow E_{obs} = E_{em}e^{-Kr}$$

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} \xrightarrow{\lambda = \frac{hc}{E}} z = \frac{E_{em} - E_{obs}}{E_{obs}} = \frac{E_{em} - E_{em}e^{-Kr}}{E_{em}e^{-Kr}} = \frac{1 - e^{-Kr}}{e^{-Kr}}$$

$$\therefore z = e^{Kr} - 1 \rightarrow Kr = \ln(1+z)$$

$$\text{or } Kr = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

if $z \ll 1$, $\underline{z \simeq Kr}$

Hubble relation: $cz = H_0 r$

$$c(Kr) = H_0 r$$

$$\therefore K = \frac{H_0}{c} = \frac{70 \text{ Km/s Mpc}}{2.9979 \times 10^5 \text{ Km/s}} = 2.3 \times 10^{-4} \text{ Mpc}^{-1}$$

3. General equation of state $p = (\gamma - 1)\rho c^2$ $(0 < \gamma < 2)$

Find $\rho(a)$: Fluid equation: $\dot{\rho} + \frac{3\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$

$$\dot{\rho} + \frac{3\dot{a}}{a}\left(\rho + \frac{(\gamma-1)}{c^2}\rho\right) = 0$$

$$\dot{\rho} + \frac{3\dot{a}}{a}(\gamma\rho) = 0$$

$$\frac{1}{a^{3\gamma}}\frac{d}{dt}(\rho a^{3\gamma}) = 0 \rightarrow \frac{d}{dt}(\rho a^{3\gamma}) = 0$$

i.e. $\rho a^{3\gamma} = \text{const.} \rightarrow \underline{\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3\gamma} \rightarrow \rho = \frac{\rho_0}{a^{3\gamma}} \text{ where } a_0 = 1}$

Friedmann equation: $\dot{a}^2 = \frac{8\pi G\rho a^2}{3} - \cancel{Kc^2}^{K=0}$

$$\therefore \dot{a}^2 = \frac{8\pi G\rho_0 a^2}{3a^{3\gamma}} = \frac{8\pi G\rho_0 a^{2-3\gamma}}{3}$$

$$\dot{a} = \left(\frac{8\pi G\rho_0}{3}\right)^{1/2} a^{(2-3\gamma)/2}$$

$$\int_0^a \frac{da}{a} = \left(\frac{8\pi G\rho_0}{3}\right)^{1/2} \int_0^t dt$$

$$\frac{a^{-\frac{(2-3\gamma)}{2}+1}}{-\frac{(2-3\gamma)}{2}+1} = \frac{a^{\frac{3}{2}\gamma}}{\frac{3}{2}\gamma} = \left(\frac{8\pi G\rho_0}{3}\right)^{1/2} t$$

$$\therefore \underline{a = \left[\frac{3\gamma}{2}\left(\frac{8\pi G\rho_0}{3}\right)^{1/2} t\right]^{2/3\gamma}}$$

Finally, $\rho = \frac{\rho_0}{a^{3\gamma}} = \frac{\rho_0}{\left[\frac{3\gamma}{2}\left(\frac{8\pi G\rho_0}{3}\right)^{1/2} t\right]^2} = \frac{\rho_0}{\frac{9\gamma^2}{4}\left(\frac{8\pi G\rho_0}{3}\right)t^2} = \underline{\frac{1}{6\pi G}(\gamma t)^{-2}}$

4. Friedmann equation: $\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho_0 a^{-3\gamma}}{3} - \frac{Kc^2}{a^2}$

For the matter term to have the same time dependence as the $\frac{Kc^2}{a^2}$ term, the dependencies on a must be the same. Therefore $\underline{\gamma = \frac{2}{3}}$

In this case, $\frac{\dot{a}^2}{\cancel{a^2}} = \frac{8\pi G\rho_0}{\cancel{a^2}} - \frac{Kc^2}{\cancel{a^2}}$

$K<0$, so RHS is +ve $\therefore$ $\dot{a} = \sqrt{8\pi G\rho_0 - Kc^2}$

and $\underline{a = \sqrt{8\pi G\rho_0 - Kc^2}\; t}$

5. Friedmann Equation w/ final term dominating:

$$\frac{\dot{a}^2}{\cancel{a^2}} \simeq -\frac{Kc^2}{\cancel{a^2}}$$

$K<0$, so $a = \sqrt{-Kc^2}\, t$

Density of matter: $\rho = \frac{\rho_0}{a^3} = \rho_0(-Kc^2)^{-3/2}t^{-3}$

Yes, this is stable, because matter term in F.E. falls as $1/a^3$ while 'K' term falls as $1/a^2$, so K term will always dominate from now on.