Phys. 4247 - Assignment #2 Solutions

1. Acceleration equation: $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$

For radiation, $p = \frac{\rho c^2}{3}$, so $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{\cancel{3}\rho \cancel{c^2}}{\cancel{3}\cancel{c^2}}\right) = -\frac{8\pi G\rho}{3}$

At the current time: $\frac{\ddot{a}_0}{a_0} = -\frac{8\pi G\rho_0}{3} \times \frac{H_0^2}{H_0^2} = -\frac{\rho_0}{\rho_{c,0}} H_0^2 = -\Omega_0 H_0^2$

where we recall that the current critical density is $\rho_{c,0} = \frac{3H_0^2}{8\pi G}$

and define $\Omega_0 = \frac{\rho_0}{\rho_{c,0}}$

The acceleration parameter $q_0 \equiv -\frac{\ddot{a}_0}{a_0}\frac{1}{H_0^2}$, so substituting $\frac{\ddot{a}_0}{a_0} = -\Omega_0 H_0^2$

gives $\underline{q_0 = \Omega_0}$

2. Circumference: The element of circumference is $dC = a(t)\, r\, d\phi$ where r = co-moving radius.

$$\therefore \; C = \int_0^{2\pi} a(t)\, r\, d\phi = a(t)\, r \int_0^{2\pi} d\phi = 2\pi a(t) r$$

Now, must relate $a(t)r$ to the proper radius s.

$$s = a(t)\int_0^r \frac{dr}{\sqrt{1-Kr^2}} = a(t) f(r)$$

In a flat universe, $\underline{K=0}$ and $f(r) = r$, so $s = a(t) r$ and $\underline{C = 2\pi s}$.

When $\underline{K=+1}$, $f(r)=\sin^{-1}r \rightarrow s=a(t)\sin^{-1}(r) \rightarrow \frac{s}{a}=\sin^{-1}(r)$

$$\therefore r=\sin\left(\frac{s}{a}\right), \quad \text{so } \underline{C=2\pi a(t)\sin\left(\frac{s}{a}\right)}$$

When $\underline{K=-1}$, $f(r)=\sinh^{-1}r \rightarrow s=a(t)\sinh^{-1}(r) \rightarrow \frac{s}{a}=\sinh^{-1}(r)$

$$\therefore r=\sinh\left(\frac{s}{a}\right), \quad \text{so } \underline{C=2\pi a(t)\sinh\left(\frac{s}{a}\right)}$$

Area: The element of area is $dA=a^2r^2\sin\theta\, d\theta\, d\phi$

$$\therefore A=a^2r^2 2\pi\int_0^{\pi}\sin\theta\, d\theta=2\pi a^2r^2\Big[-\cos\theta\Big]_0^{\pi}=2\pi a^2r^2(1+1)$$

$$=4\pi a^2r^2$$

From above we know what r is in terms of s & a for each K.

$$\therefore K=0, \quad A=4\pi s^2$$
$$=1, \quad A=4\pi a^2\sin^2\left(\frac{s}{a}\right)$$
$$=-1, \quad A=4\pi a^2\sinh^2\left(\frac{s}{a}\right)$$

Volume: The element of volume is $dV=\dfrac{a^3r^2\sin\theta\, d\theta\, d\phi\, dr}{\sqrt{1-Kr^2}}$

$$\therefore V=4\pi a^3\int_0^{r}\frac{r^2dr}{\sqrt{1-Kr^2}}$$

When $\underline{K=0}$, $V=\frac{4}{3}\pi a^3r^3$, but $s^3=a^3r^3$, so $\underline{V=\frac{4}{3}\pi s^3}$

When $K=+1$, $V=4\pi a^3\displaystyle\int_0^{r}\frac{r^2dr}{\sqrt{1-r^2}}=4\pi a^3\left[-\frac{r}{2}\sqrt{1-r^2}+\frac{1}{2}\sin^{-1}(r)\right]_0^{r}$

$$\therefore V = 4\pi a^3\left[-\frac{r}{2}\sqrt{1-r^2} + \frac{1}{2}\sin^{-1}(r)\right]$$

From above, when $\underline{K=+1,}$ $r = \sin\left(\frac{s}{a}\right)$

$$\therefore V = 4\pi a^3\left[\frac{-\sin\left(\frac{s}{a}\right)\cos\left(\frac{s}{a}\right)}{2} + \frac{s}{2a}\right] = 4\pi a^3\left(\frac{s}{2a} - \frac{1}{4}\sin\left(\frac{2s}{a}\right)\right)$$

$$= \underline{2\pi a^3\left(\frac{s}{a} - \frac{1}{2}\sin\left(\frac{2s}{a}\right)\right)}$$

When $K=-1$, $V = 4\pi a^3 \int_0^r \frac{r^2\,dr}{\sqrt{1+r^2}} = 4\pi a^3\left[\frac{1}{2}\left(r\sqrt{r^2+1} - \sinh^{-1}(r)\right)\right]_0^r$

$$= 2\pi a^3\left(r\sqrt{r^2+1} - \sinh^{-1}(r)\right)$$

From above, when $\underline{K=-1,}$ $r = \sinh\left(\frac{s}{a}\right)$

$$\therefore V = 2\pi a^3\left(\sinh\left(\frac{s}{a}\right)\cosh\left(\frac{s}{a}\right) - \frac{s}{a}\right) = \underline{2\pi a^3\left(\frac{\sinh\left(\frac{2s}{a}\right)}{2} - \frac{s}{a}\right)}$$

3. No. Closed & open correspond to the geometry of space-time and is set by the total energy content of the Universe.

Consider $(\Omega_m + \Omega_\Lambda - 1) = \frac{Kc^2}{a^2H^2}$

sign of RHS is set by K (a constant) so is fixed.

$\therefore$ sign of LHS is fixed.

4. Static universe implies both $\dot{a}$ and $\ddot{a}$ are zero.

$$\therefore \quad 0 = \frac{8\pi G\rho}{3} - \frac{Kc^2}{a^2} + \frac{\Lambda}{3} \quad \text{and} \quad 0 = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3} \quad (\text{note } P=0)$$

$\therefore \frac{\Lambda}{3} = \frac{4\pi G\rho}{3}$. Since $\rho > 0$ (its matter), $\underline{\Lambda > 0}$

From the Friedmann equation, $0 = \frac{2\Lambda}{3} - \frac{Kc^2}{a^2} + \frac{\Lambda}{3}$

$\rightarrow \Lambda = \frac{Kc^2}{a^2} > 0$, so $K > 0$ $\therefore$ <u>a closed Universe</u>

Unstable, because $\rho \propto \frac{1}{a^3}$ and $\frac{\Lambda}{3}$ = constant, so if a increases even a tiny bit, ρ drops and acc'n dominates.

5. Recall $\Omega_m = \frac{\rho_m}{\rho_c} = \frac{8\pi G\rho_m}{3H^2}$ and $\Omega_{m,0} = \frac{\rho_0}{\rho_c} = \frac{8\pi G\rho_0}{3H_0^2}$

Friedmann equation: $\dot{a}^2 = \frac{8\pi G\rho_m a^2}{3} - Kc^2$ $\quad (\Lambda = 0)$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_m - \frac{Kc^2}{a^2} \rightarrow \frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho_0 a_0^3}{3a^3} - \frac{Kc^2}{a^2}\frac{a_0^2}{a_0^2} \text{ since } \rho_m = \rho_0\left(\frac{a_0}{a}\right)^3$$

But $\frac{a_0}{a} = (1+z)$ and $\frac{Kc^2}{a_0^2} = H_0^2(\Omega_{m,0} - 1)$.

$$\therefore \text{F.E. is } \frac{\dot{a}^2}{a^2} = H^2 = \frac{8\pi G\rho_0(1+z)^3}{3}\frac{H_0^2}{H_0^2} - H_0^2(\Omega_{m,0}-1)(1+z)^2$$

$$\text{or, } H^2 = \Omega_{m,0}H_0^2(1+z)^3 - H_0^2(\Omega_{m,0}-1)(1+z)^2$$

$$= H_0^2(1+z)^2\left(\Omega_{m,0}(1+z) - (\Omega_{m,0}-1)\right) = H_0^2(1+z)^2\left[\cancel{\Omega_{m,0}} + \Omega_{m,0}z - \cancel{\Omega_{m,0}} + 1\right]$$

$$H^2 = H_0^2(1+z)^2(1+\Omega_{m,0}z)$$

Sub. this into def'n of $\Omega_m(z)$ and use $\rho_m = \rho_0\left(\frac{a_0}{a}\right)^3$ again

So $\Omega_m = \dfrac{8\pi G \rho_m}{3H_0^2(1+z)^2(1+\Omega_{m,0}z)} = \dfrac{8\pi G \rho_0}{3H_0^2(1+z)^2(1+\Omega_{m,0}z)} \dfrac{a_0^3}{a^3}$

$$\therefore \quad \Omega_m(z) = \frac{\Omega_{m,0}(1+z)^{\cancel{3}}}{\cancel{(1+z)^2}(1+\Omega_{m,0}z)} = \underline{\frac{\Omega_{m,0}(1+z)}{(1+\Omega_{m,0}z)}}$$