Assignment #3 - Phys. 4247 Solutions

#1. a) For spatially-flat matter dominated cosmologies with a cosmological constant the F.E is:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho_{matt}\, a^2 + \frac{\Lambda a^2}{3}$$

$$= \frac{8\pi G}{3}\left(\rho_{matt,o}\frac{a_o^3}{a^3}\right)a^2 + \frac{\Lambda a^2}{3H_o^2}H_o^2$$

$$= \frac{8\pi G \rho_{matt,o}}{3H_o^2}(1+z)^3 a^2 H_o^2 + \Omega_{\Lambda,o} H_o^2 a^2$$

$$= \Omega_{m,o}(1+z)^3 a^2 H_o^2 + \Omega_{\Lambda,o} H_o^2 a^2$$

Since $K=0$, we know that $1-\Omega_o = 1-(\Omega_{m,o}+\Omega_{\Lambda,o}) = 0$

$$\therefore \Omega_{\Lambda,o} = 1-\Omega_{m,o}$$

So, $\dot{a}^2 = \Omega_{m,o}(1+z)^3 a^2 H_o^2 + (1-\Omega_{m,o})a^2 H_o^2$

$$\therefore \frac{\dot{a}^2}{a^2} = \left(\frac{\dot{a}}{a}\right)^2 = H^2 = \underline{H_o^2\left(\Omega_{m,o}(1+z)^3 + 1 - \Omega_{m,o}\right)}$$

b) Co-ordinate or co-moving distance to redshift z

$$\int_0^{r_o}\frac{dr}{\sqrt{1-Kr^2}} \overset{\text{flat Univ.}}{=} r_o = c\int_t^{t_o}\frac{dt}{a(t)} \quad \text{from the R-W metric}$$

Coordinate transformations: $\frac{da}{dt} = \dot{a} \rightarrow dt = \frac{da}{\dot{a}}$

$$\frac{a_o}{a} = (1+z) \rightarrow z = \frac{a_o}{a} - 1 \rightarrow \frac{dz}{da} = \frac{-a_o}{a^2} = -\frac{(1+z)}{a}$$

$$\therefore da = \frac{-a\,dz}{(1+z)} \rightarrow dt = \frac{-a\,dz}{\dot{a}(1+z)} = \frac{-dz}{H(z)(1+z)}$$

$$\therefore \quad r_0 = c\int_z^0 \frac{-dz}{H(z)(1+z)}\frac{1}{a} = c\int_0^z \frac{dz}{H(z)} = c\int_0^z \frac{dz}{H_0\left[1-\Omega_{m,0}+\Omega_{m,0}(1+z)^3\right]^{1/2}}$$

$$\frac{1}{a} = (1+z)$$

or $$r_0 = cH_0^{-1}\int_0^z \frac{dz}{\left[1-\Omega_{m,0}+\Omega_{m,0}(1+z)^3\right]^{1/2}}$$

c) Set $\Omega_{m,0}=1$. Then $r_0 = cH_0^{-1}\int_0^z \frac{dz}{(1+z)^{3/2}} = cH_0^{-1}\left[\frac{-2}{\sqrt{1+z}}\right]_0^z$

$$= cH_0^{-1}\left(\frac{-2}{\sqrt{1+z}} + 2\right) = 2cH_0^{-1}\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

$$d_L = r_0 a_0 (1+z) = 2cH_0^{-1}(1+z)\left(1-\frac{1}{\sqrt{1+z}}\right) = 6000h^{-1}(1+z)\left[1-\frac{1}{\sqrt{1+z}}\right] \text{ Mpc}$$

$$d_A = \frac{r_0 a_0}{(1+z)} = \frac{d_L}{(1+z)^2} = \frac{6000h^{-1}}{(1+z)}\left[1-\frac{1}{\sqrt{1+z}}\right] \text{Mpc.}$$

2.a) From our derivation of cosmological redshift we found that time intervals at different scale factors are related by

$$\frac{\Delta t_e}{\Delta t_0} = \frac{a(t_e)}{a(t_0)} = \frac{1}{1+z}$$

$$\therefore \quad \Delta t_e = \frac{\Delta t_0}{(1+z)} = \frac{3 \text{ days}}{6} = 0.5 \text{ days}$$

b) $$R_{max} = c\Delta t_e = \left(2.9979\times10^8 \tfrac{m}{s}\right)(0.5 \text{ days})\left(\frac{86400 s}{day}\right)\left(\frac{1 pc}{3.086\times10^{16} m}\right)$$

$$= 4.2\times10^{-4} \text{ pc}$$

c) $\theta = \frac{R_{max}}{d_A} = \frac{4.2\times10^{-4}\,pc\,(1+z)}{r_0 a_0}$

For a matter dominated Universe w/ $\Omega_{m,0}=0.3$, use the Mattig formula:

$$r_0 = \frac{4c}{H_0 a_0}\frac{1}{\Omega_{m,0}^2}\frac{1}{(1+z)}\left[\frac{\Omega_{m,0}z}{2} + \left(1-\frac{1}{2}\Omega_{m,0}\right)\left(1-(1+\Omega_{m,0}z)^{1/2}\right)\right]$$

$$\therefore \theta = \frac{(4.2\times10^{-4}\,pc)\,\Omega_{m,0}^2\,(1+z)^2}{4\left(\frac{c}{H_0}\right)\left[\frac{\Omega_{m,0}z}{2} + \left(1-\frac{1}{2}\Omega_{m,0}\right)\left(1-(1+\Omega_{m,0}z)^{1/2}\right)\right]}$$

Subistitute in $z=5$, $\Omega_{m,0}=0.3$; $H_0 = 70$ Km/s Mpc^{-1},

$d_A = 1353.3$ Mpc

$$\therefore \theta = \frac{4.2\times10^{-4}\,pc}{1353.3\times10^6\,pc} = 3.1\times10^{-13} \text{ radians}\left(\frac{180 \text{ degrees}}{\pi \text{ radians}}\right)\left(\frac{3600 \text{ arcsec}}{1 \text{ degree}}\right)$$

$$= \underline{6.4\times10^{-8} \text{ arc seconds}}$$

3. For a $K=0$, $\Lambda=0$ Universe, the F.E is

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3}$$

Know that when $K=0$, $a\propto t^{2/3}$ for matter. Write $a=\alpha t^{2/3}$

So $\dot{a} = \frac{2}{3}\alpha t^{-1/3}$ and $\dot{a}^2 = \frac{4}{9}\alpha^2 t^{-2/3}$

$$\therefore \frac{\dot{a}^2}{a^2} = \frac{\frac{4}{9}\alpha^2 t^{-2/3}}{\alpha^2 t^{4/3}} = \frac{4}{9}t^{-2} = \frac{\cancel{4}}{{}_3 9t^2} = {}^{3}\frac{8\pi G\rho}{\cancel{3}}$$

$$\therefore \underline{\rho = \frac{1}{6\pi G t^2}}$$

4. Proper distance is defined as

$$d_{prop} = a_0 \int_0^{r_1} \frac{dr}{\sqrt{1-Kr^2}}$$

From propogation of light equation

$$\int_0^{r_1} \frac{dr}{\sqrt{1-Kr^2}} = c \int_{t_{min}}^{t_0} \frac{dt}{a(t)}$$ where t_{min} is the time

light is emitted from the furthest galaxy.

$$\therefore \underline{d_{prop} = ca_0 \int_{t_{min}}^{t_0} \frac{dt}{a(t)}}$$

5. Assuming the half-light radius = half-mass radius, the stars are on isotropic orbits, and that the measured velocity dispersion does not change rapidly with radius, estimate the total mass of the galaxy to be

$$M = \frac{\langle v^2 \rangle r_h}{\alpha G} = \frac{3\sigma^2 (120_{pc})}{(0.4)\, G}$$

$$\therefore M = \frac{3}{0.4G} (10.5)^2 \frac{Km^2}{s^2} \left(\frac{(1000)^2 m^2}{1\ Km^2}\right) 120pc \left(\frac{3.086 \times 10^{16} m}{1\ pc}\right)$$

$$= 4.59 \times 10^{37} g = \underline{2.31 \times 10^7 M_\odot}$$

$$\therefore \frac{M}{L} = \frac{2.31 \times 10^7 M_\odot}{1.8 \times 10^5 L_\odot} = \underline{128}$$

Dwarf galaxies are more dark-matter 'rich' than large spiral galaxies, but less than galaxy clusters. Give the large numbers of dwarf galaxies, then these are considered good targets for dark matter detection using γ-rays.