# Phys 4247 - Assignment #4 Solutions

1. Observable Universe constitutes a sphere with proper radius $d_h = \frac{2c}{H_0}$ for a flat Universe. Since we're dealing with flat geometry, the observable volume is $V = \frac{4}{3}\pi\left(\frac{2c}{H_0}\right)^3 = \frac{4}{3}\pi\left(\frac{2c}{100\frac{km}{s\,Mpc}h}\right)^3$

$$\therefore V = 9\times10^{11}\, h^{-3}\, Mpc^3$$

Total baryonic mass in the observable Universe:

$$M_b = \Omega_b \rho_{crit} V = 0.046\times\left(\frac{2.78\, h^{-1}\times10^{11} M_\odot}{h^{-3}\, Mpc^3}\right)\left(9\times10^{11}\, h^{-3}\, Mpc^3\right)$$

$$= 1.15\times10^{22}\, h^{-1} M_\odot$$

If a typical galaxy has $10^{11} M_\odot$ of baryonic matter and $h = 0.7$, then the # of galaxies in the observable Universe is $N \simeq \frac{1.15\times10^{22}(0.7)^{-1} M_\odot}{10^{11}\, M_\odot/\text{galaxy}} \simeq 1.6\times10^{11}$ galaxies

Total number of protons. 24% of mass is $^4He$, rest is $H$.

$$\therefore \#\text{ of protons} = 0.24\left(1.15\times10^{22}\, h^{-1} M_\odot\right)\left(\frac{1.989\times10^{30}\,Kg}{1\, M_\odot}\right)\left(\frac{1\ {}^4He}{4\times1.67\times10^{-27}\,Kg}\right)\left(\frac{2\text{ protons}}{1\ {}^4He}\right)$$

$$+0.76\left(1.15\times10^{22}\, h^{-1} M_\odot\right)\left(\frac{1.989\times10^{30}\,Kg}{1\, M_\odot}\right)\left(\frac{1\ H}{1.67\times10^{-27}\,Kg}\right)\left(\frac{1\text{ proton}}{1\ H}\right)$$

$= 1.2\times10^{79}\, h^{-1}$ protons or $1.7\times10^{79}$ protons if $h = 0.7$

2\. a) Density parameter $\Omega(t) = \dfrac{\rho(t)}{\rho_c(t)} = \dfrac{\rho(t)\, 8\pi G}{3H^2(t)} = \dfrac{\rho_0 a^{-3}\, 8\pi G}{3H^2}$

b/c for matter $\rho(t) = \rho_0 a^{-3}$ (taking $a_0 = 1$)

$$\therefore \; \Omega(t) = \frac{\rho_0\, 8\pi G}{3H_0^2} \frac{a^{-3} H_0^2}{H^2} = \frac{\rho_0}{\rho_{c,0}} \frac{a^{-3} H_0^2}{H^2} = \Omega_{m,0} \frac{a^{-3} H_0^2}{H^2}$$

but $(1+z) = \frac{1}{a}$ if $a_0 = 1$, so $\Omega(z) = \dfrac{\Omega_{m,0} (1+z)^3 H_0^2}{H^2}$

$$\therefore \; \underline{\frac{H^2 \Omega_m(z)}{(1+z)^3} = \text{constant} = \Omega_{m,0} H_0^2}$$

b) The Hubble parameter at decoupling as a fraction of its current value is $\dfrac{H(z_d)}{H_0} = \dfrac{\Omega_{m,0}^{1/2} (1+z_d)^{3/2}}{\Omega_m^{1/2}} \simeq \Omega_{m,0}^{1/2} (1+z_d)^{3/2}$

where $z_d = 1100$ is the redshift at decoupling and $\Omega_m(z_d) \simeq 1$

Take $\Omega_{m,0} = 0.3 \; \therefore \; \underline{\dfrac{H(z_d)}{H_0} = 2.0 \times 10^4}$

c) In Assignment #3, Problem #1b, we found that the co-moving distance in a flat Universe with a cosmological constant is

$$r_0 = cH_0^{-1} \int_0^z \frac{dz}{\left[1 - \Omega_{m,0} + \Omega_{m,0}(1+z)^3\right]^{1/2}}$$

When $z \gg 1$, this becomes $r_0 = cH_0^{-1} \displaystyle\int_0^z \frac{dz}{\Omega_{m,0}^{1/2} (1+z)^{3/2}} = \frac{cH_0^{-1}}{\sqrt{\Omega_{m,0}}} \left[\frac{-2}{\sqrt{1+z}}\right]_0^z$

$$= \frac{cH_0^{-1}}{\sqrt{\Omega_{m,0}}} \left(\frac{-2}{\sqrt{1+z}} + 2\right) \simeq \underline{\frac{2cH_0^{-1}}{\sqrt{\Omega_{m,0}}}} \quad \text{if } z \gg 1$$

d) Angle subtended by Hubble length at decoupling is

$$\Theta = \frac{c H(z_d)^{-1}}{d_A(z_d)} = \frac{\cancel{c}\, \Omega_{m,0}^{-1/2} (1+z_d)^{-3/2} \cancel{H_0^{-1}}\, \cancel{\Omega_{m,0}^{1/2}} (1+z_d)}{2 \cancel{c} \cancel{H_0^{-1}}}$$

using the results of parts (b) and (c)

$$\therefore \Theta \simeq \frac{1}{2}(1+z_d)^{-1/2} = \frac{1}{2}(1101)^{-1/2} = 0.015 \text{ rad} \left(\frac{360 \text{ deg}}{2\pi \text{ rad}}\right) \simeq \underline{0.9 \text{ deg}}$$

3. a) The Mattig formula for an open Universe is

$$r_0 = \frac{4c}{H_0} \frac{1}{\Omega_{m,0}^2} \frac{1}{(1+z)} \left[ \frac{\Omega_{m,0} z}{2} + \left(1 - \frac{\Omega_{m,0}}{2}\right)\left(1 - (1+\Omega_{m,0} z)^{1/2}\right) \right]$$

∴ The angular diameter distance to $z$ is

$$d_{ang} = \frac{4c}{H_0} \frac{1}{\Omega_{m,0}^2 (1+z)^2} \left[ \frac{\Omega_{m,0} z}{2} + \left(\frac{2 - \Omega_{m,0}}{2}\right)\left(1 - (1+\Omega_{m,0} z)^{1/2}\right) \right]$$

$$= 2cH_0^{-1} \frac{\left(\Omega_{m,0} z + (-1)(\Omega_{m,0} - 2)(-1)\left((1+\Omega_{m,0} z)^{1/2} - 1\right)\right)}{\Omega_{m,0}^2 (1+z)^2};$$

$$\therefore \underline{d_{ang} = 2cH_0^{-1} \frac{\left(\Omega_{m,0} z + (\Omega_{m,0} - 2)\left((1+\Omega_{m,0} z)^{1/2} - 1\right)\right)}{\Omega_{m,0}^2 (1+z)^2}}$$

b) To use this distance to compute the angular size of the Hubble length at decoupling, split into 2 terms and consider how they behave as $z$ gets large.

$$d_{ang} = 2cH_0^{-1} \left[ \frac{\Omega_{m,0} z}{\Omega_{m,0}^2 (1+z)^2} + \frac{(\Omega_{m,0} - 2)(\sqrt{1+\Omega_{m,0} z} - 1)}{\Omega_{m,0}^2 (1+z)^2} \right]$$

The first term falls as $\sim 1/z$ as $z$ gets large. The $2^{nd}$ term falls as $\sim 1/z^{3/2}$ as $z$ gets large. So at decoupling the $1^{st}$ term will dominate and

$$d_{Ang} \simeq \frac{2cH_0^{-1}z}{\Omega_{m,0}(1+z)^2} \simeq \frac{2cH_0^{-1}(1+z)}{\Omega_{m,0}(1+z)^2} = \frac{2cH_0^{-1}}{\Omega_{m,0}(1+z)}$$

So, $$\theta = \frac{cH(z_d)^{-1}}{d_{Ang}} = \frac{c\,\Omega_{m,0}^{-1/2}(1+z_d)^{-3/2}H_0^{-1}\,\Omega_{m,0}(1+z)}{2cH_0^{-1}}$$

$$= \frac{1}{2}(1+z_d)^{-1/2}\Omega_{m,0}^{1/2} \simeq (1\,deg)\,\Omega_{m,0}^{1/2}$$ using the result of 2(d)

If $\Omega_{m,0}=1$, $\ell \simeq 220$, then this implies

$$\theta \sim \frac{220^\circ}{\ell}$$

So if $\Omega_{m,0}=0.3$, $\theta \simeq 0.55\,deg \rightarrow \ell \sim 400$

A peak at $\ell \sim 400$ is completely incompatable w/ the data.

4. Energy emitted by starlight over 10 Gyr:

$$\text{Energy} = Lt = (2.3\times10^{10}L_\odot)\left(\frac{3.85\times10^{26}\,W}{1\,L_\odot}\right)\left(\frac{3.16\times10^{7}\,s}{1\,yr}\right)10\times10^{9}\,yr$$

$$= \underline{2.8\times10^{54}\,J}$$

If fusion of $^4He$ produces 28.4 MeV and all the energy in the starlight is generated by fusion then

$$\#\text{ of }^4He\text{ atoms created} = (2.8\times10^{54}\,J)\left(\frac{1\,eV}{1.602\times10^{-19}\,J}\right)\left(\frac{1\ ^4He\text{ created}}{28.4\times10^{6}\,eV}\right) = \underline{6.2\times10^{65}}$$

Mass in $^4He$ created $= 6.2\times10^{65}$ $^4He$ atoms $\left(\frac{4\times1.67\times10^{-27}\,Kg}{1\,^4He}\right) = 4.1\times10^{39}\,Kg = 2.1\times10^{9}\,M_\odot$

Orignal mass in $^4He$ using the primordial mass fraction

$$= Y_4 \times 10^{11} M_\odot = (0.24)(10^{11} M_\odot) = 2.4\times10^{10} M_\odot$$

New mass in $^4He = 2.4\times10^{10} M_\odot + 2.1\times10^{9} M_\odot = 2.6\times10^{10} M_\odot$

New $Y_4 = \dfrac{2.6\times10^{10} M_\odot}{10^{11} M_\odot} = 0.26$, so the $^4He$ fraction has increased by 0.02 over its primordial value.

5. The cross-section is $\sigma_W \sim 10^{-47}\left(\frac{5\times10^{-4}\,eV}{10^6\,eV}\right)^2 m^2 = 2.5\times10^{-66}\,m^2$

The mean free path of this low energy neutrino through $^{56}Fe$ is $l = \dfrac{1}{n\sigma_W}$ where $n$ is the number density of particles in the lead.

$$\#\text{ density of atoms in } ^{56}Fe = \frac{\rho_{^{56}Fe}}{M_{nucleus}} = \frac{7900\,Kg\,m^{-3}}{55.8\times(1.66\times10^{-27}\,Kg)} = 8.5\times10^{28}\,m^{-3}$$

$$\#\text{ density of particles in } ^{56}Fe = (26\text{ protons})\,8.5\times10^{28}\,m^{-3} + (30\text{ neutrons})\,8.5\times10^{28}\,m^{-3} + (26\text{ electrons})\,8.5\times10^{28}\,m^{-3} = 7\times10^{30}\,m^{-3}$$

$$\therefore\ l = \frac{1}{(7\times10^{30}\,m^{-3})(2.5\times10^{-66}\,m^2)} = 5.7\times10^{34}\,m = 1.8\times10^{12}\,Mpc \gg \frac{c}{H_0}\ !!$$