Comptonization (cont.)

A photon spectrum interacting w/ this plasma will be modified b/c of repeated upscatterings & downscatterings that depend on the kinematics of the individual collisions. While the individual shifts in energy are very small, the accumulated effect can give rise to substantial deformation.

To determine if Comptonization is likely to be important, use the Compton y parameter:

$$y = \begin{bmatrix} \text{avg. fractional} \\ \text{energy shift per scattering} \end{bmatrix} \times \begin{pmatrix} \text{mean \# of} \\ \text{scatterings} \end{pmatrix}$$

Lets consider the interaction of an isotropic photon field & a Maxwellian e^- dist'n w/ temp. T. Work in the Thomson limit.

Recall that a rel. e^- loses energy at a rate $\frac{dE}{dt} = \frac{4}{3}\sigma_T c U_{ph} \beta^2 \gamma^2$

Turn this around & think of $-\frac{dE}{dt}$ as the energy gained by the initial photon population. The # of scatterings/time: $\frac{\sigma_T c U_{ph}}{\epsilon}$

Then the avg. energy gained by photons/scatter is

$$\left\langle \frac{\Delta \epsilon}{\epsilon} \right\rangle_+ = \frac{4}{3}\beta^2$$

where $\gamma \gg 1$ & avg. is over all angles

Since this is a Maxwellian dist'n of e^-, $\langle\beta^2\rangle = \frac{3kT}{mc^2}$, so

$$\left\langle \frac{\Delta\epsilon}{\epsilon} \right\rangle_+ = \frac{4kT}{mc^2}$$

From this gain term, we need to subtract an energy loss term that accounts for e^- recoil. From before, we saw that $\epsilon_1 \simeq \gamma^2\epsilon\left(1+\frac{\gamma\epsilon}{mc^2}\right)^{-1} \simeq \epsilon\left(1-\frac{\epsilon}{mc^2}\right) = \epsilon - \frac{\epsilon^2}{mc^2}$

$\gamma = 1$; $\epsilon \ll mc^2$

$$\rightarrow \left\langle \frac{\epsilon_1 - \epsilon}{\epsilon} \right\rangle = -\frac{\epsilon}{mc^2}$$

Thus the net energy exchanged in a collision is approx.

$$\left\langle \frac{\Delta\epsilon}{\epsilon} \right\rangle = \frac{4kT - \epsilon}{mc^2}$$

$\therefore$ On avg., photons gain energy whenever $4kT > \epsilon$ and lose energy otherwise.

The mean number of scatterings depends on the optical depth. Random walk considerations $N \simeq \tau^2$ $(\tau \gg 1)$; $N \simeq 1 - e^{-\tau}$ $(\tau \ll 1$; $N \simeq \tau)$

So, can estimate N by $N \simeq \max(\tau, \tau^2)$

$\therefore$ The Compton y-parameter is $y = \frac{4kT}{mc^2}\left(1 - \frac{\epsilon}{4kT}\right)\mathrm{Max}(\tau, \tau^2)$

which is usually written as $y = \frac{4kT}{mc^2}\mathrm{Max}(\tau, \tau^2)$

b/c $4kT \gg \epsilon$

Example:

Imagine an isothermal medium of temp. T w/ $\Upsilon > 1$. Assume that a photon w/ $\epsilon_0 << kT$ is injected into the medium and suffers multiple scatters. How does the energy of the emergent photon depend on Υ?

Net fractional change/collision: $\left\langle \frac{\Delta\epsilon}{\epsilon} \right\rangle = \frac{4kT - \epsilon}{mc^2} = \frac{4kT}{mc^2} - \frac{\epsilon}{mc^2}$

or $\frac{d\epsilon}{dN} = \left(\frac{4kT}{mc^2} - \frac{\epsilon}{mc^2} \right)\epsilon$ where N is the # of scatterings & is treated as a continuous variable

Define a dimensionless constant $A = \frac{4kT}{mc^2}$; $\epsilon' = \frac{\epsilon}{mc^2}$

then $\frac{d\epsilon'}{dN} = A\epsilon' - \epsilon'^2$

$$\int_{\epsilon_0'}^{\epsilon'} \frac{d\epsilon'}{A\epsilon' - \epsilon'^2} = \int_0^N dN$$

$$\int_{\epsilon_0'}^{\epsilon'} \frac{d\epsilon'}{A\epsilon' + (-1)\epsilon'^2} = N$$

$$-\frac{1}{A} \ln\left(\frac{A + (-1)\epsilon'}{\epsilon'} \right) \Bigg]_{\epsilon_0'}^{\epsilon'} = N$$

$$\ln\left(\left(\frac{A - \epsilon'}{\epsilon'}\right) \Big/ \left(\frac{A - \epsilon_0'}{\epsilon_0'}\right) \right) = -AN$$

$$\left(\quad \epsilon' \Big/ \left(\frac{A-\epsilon_0'}{\epsilon_0'}\right)\right) = -AN$$

$$\frac{\dfrac{A-\epsilon'}{\epsilon'}}{\dfrac{A-\epsilon_0'}{\epsilon_0'}} = e^{-AN}$$

$$\frac{e^{AN}\epsilon_0'}{\underbrace{(A-\epsilon_0')}_{\Rightarrow\ \simeq A \text{ by assumptions}}} = \frac{\epsilon'}{A-\epsilon'}$$

Set $y = AN$

$$\therefore \quad \frac{e^{y}\frac{\epsilon_0}{mc^2}}{A} = \frac{\frac{\epsilon}{mc^2}}{A - \epsilon/mc^2}$$

... more algebra to solve for ϵ...

$$\epsilon = \frac{\epsilon_0 e^{y}}{\left(1 + e^{y}\left(\frac{\epsilon_0}{4KT}\right)\right)}$$

Recall that $\epsilon_0 \ll KT$, so if y is small, $\epsilon \simeq \epsilon_0 e^{y}$

if y is large, $\epsilon \to 4KT$ and no further energy exchange occurs

$\to$ in this $y \gg 1$ limit, Comptonization is 'saturated'

Also, the ϵ is independent of ϵ_0. The photon dist'n forgets its initial conditions.

To quantify the saturation of Comptonization, define a critical y parameter, y_{crit}, such that a photon has attained ½ of its energy at saturation, ie, $\epsilon = 2kT$ when $y = y_{crit}$

$$2kT = \frac{\epsilon_0 e^{y_{crit}}}{1 + e^{y_{crit}}\left(\frac{\epsilon_0}{4kT}\right)}$$

$$2kT + \frac{e^{y_{crit}}\epsilon_0}{2} = \epsilon_0 e^{y_{crit}}$$

$$2kT = e^{y_{crit}}\left(\frac{\epsilon_0}{2}\right)$$

$$\therefore y_{crit} = \ln\left(\frac{4kT}{\epsilon_0}\right)$$

In terms of a critical e^- scattering optical depth, τ_{crit}, where we assume $N \sim \tau^2$

$$\tau_{crit} = \left(\frac{mc^2}{4kT}\ln\left(\frac{4kT}{\epsilon_0}\right)\right)^{1/2}$$

For a given kT ; ϵ_0, gives an estimate of the τ at which saturation begins to occur.

e.g. optical photon ($\epsilon_0 \sim 4\,eV$) into a gas w/ $kT \sim 10$ keV, then $\tau_{crit} \sim 10$, ie after 100 scatterings $\epsilon_0 \to \sim 4kT$