Bondi-Hoyle Accretion

Consider a gravitating mass M accreting spherically symmetrically from a large gas cloud. Ignore B-field, ang. mom. & any bulk motion of the gas. Given the density $\rho(\infty)$ & temperature $T(\infty)$ far from M, what is the accretion rate $\dot{M}$ & what is the radius of the accreting sphere?

Solve w/ the equ's of hydrodynamics in spherical coordinates and in steady state.

- only need the radial coord's
- assume $v = v_r < 0$ for infall

Mass conservation (continuity eq'n.): $\cancel{\frac{\partial \rho}{\partial t}}^{\,0} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$

$$\frac{1}{r^2} \frac{d}{dr} (r^2 \rho v) = 0$$

$$\rightarrow r^2 \rho v = \text{constant}$$

But $\rho(-v)$ is the mass flux through a spherical shell

or $4\pi r^2 \rho(-v) = \dot{M}$ at radius r

$\therefore \dot{M}$ is the same at all r

Momentum conservation (Euler eqn.): $\cancel{\rho \frac{d\vec{v}}{dt}}^{\,0} + \underbrace{\rho \vec{v} \cdot \vec{\nabla} \vec{v}}_{\text{cons. of mom.}} = -\underbrace{\vec{\nabla} P}_{\text{Press. grad.}} + \overbrace{\vec{f}}^{\text{ext. force/vol.}}$

cons. of mom. by vel grad.

Press. grad.

in our case $|\vec{f}| = -\frac{GM\rho}{r^2}$

$$\Rightarrow \quad v\frac{dv}{dr} + \frac{1}{\rho}\frac{dP}{dr} + \frac{GM}{r^2} = 0$$

Eqn. of state of gas: $P = K\rho^\gamma$, $K = \text{const.}$

γ = adiabatic index, or ratio of specific heats
= 1 for isothermal gas
= 5/3 for adiabatic gas

To get the temperature use the ideal gas law: $T = \frac{\mu m_H P}{\rho k}$

μ = mean molecular weight
= 1 for H gas
= $\frac{1}{2}$ for fully ionized H

Now $\frac{dP}{dr} = \frac{dP}{d\rho}\frac{d\rho}{dr} = c_s^2 \frac{d\rho}{dr}$ where $c_s = \sqrt{\left(\frac{dP}{d\rho}\right)}$ is the sound speed.

Now, Euler eqn is $v\frac{dv}{dr} + \frac{1}{\rho}c_s^2\frac{d\rho}{dr} + \frac{GM}{r^2} = 0$

But, from continuity $\frac{1}{r^2}\frac{d}{dr}(r^2\rho v) = \frac{1}{r^2}\left[\frac{d\rho}{dr}\cdot r^2 v + \rho\frac{d}{dr}(r^2 v)\right]$

$$= v\frac{d\rho}{dr} + \frac{\rho}{r^2}\frac{d}{dr}(r^2 v) = 0$$

$$\therefore \frac{1}{\rho}\frac{d\rho}{dr} = -\frac{1}{vr^2}\frac{d}{dr}(r^2 v)$$

Sub. into Euler:

$$v\frac{dv}{dr} - \frac{c_s^2}{vr^2}\frac{d(r^2 v)}{dr} + \frac{GM}{r^2} = 0$$

$$\frac{1}{2}\frac{d(v^2)}{dr} - \frac{c_s^2}{vr^2}\left(2rv + r^2\frac{dv}{dr}\right) = -\frac{GM}{r^2}$$

$$\frac{1}{2}\frac{d(v^2)}{dr} - \frac{2c_s^2}{r} - \frac{c_s^2}{v}\frac{dv}{dr} = -\frac{GM}{r^2}$$

$$\frac{1}{2}\frac{d(v^2)}{dr} - \frac{1}{2}\frac{c_s^2}{v^2}\frac{d(v^2)}{dr} = -\frac{GM}{r^2} + \frac{2c_s^2}{r^2}$$

$$\frac{1}{2}\left(1 - \frac{c_s^2}{v^2}\right)\frac{d(v^2)}{dr} = -\frac{GM}{r^2}\left(1 - \frac{2c_s^2 r}{GM}\right)$$

Analyze: at large r: c_s must be finite while $r \to \infty$, so RHS +ve

$\frac{d(v^2)}{dr} < 0$ since we want $v^2 \uparrow$ as $r \downarrow$

$\therefore (1 - c_s^2/v^2) < 0 \quad \therefore v^2 < c_s^2$

ie gas is subsonic at large r

at small r: RHS -ve since c_s/T would have to be v. large to compensate

$\frac{d(v^2)}{dr} < 0$ always

$\therefore \left(1 - \frac{c_s^2}{v^2}\right) > 0 \Rightarrow v^2 > c_s^2$

ie gas is supersonic close to the origin.

The flow is called <u>transonic</u> and passes through a sonic or critical point r_s where $v(r_s) = c_s(r_s)$

Clearly, this happens when $1 - \frac{2c_s^2(r_s)r_s}{GM} = 0$

$$\text{or} \quad r_s = \frac{GM}{2c_s^2(r_s)}$$