Now that we understand some details, lets solve it!

- integrate Euler eqn over radius

$$\frac{1}{2}v^2 + \int \frac{dP}{\rho} - \frac{GM}{r} = \text{constant}$$

From our e.o.s. $dP = K\gamma \rho^{\gamma-1} d\rho$

so $$\frac{1}{2}v^2 + K\gamma \int \frac{\rho^{\gamma-1}}{\rho} d\rho - \frac{GM}{r} = \text{const.}$$

$$\frac{1}{2}v^2 + \frac{K\gamma}{\gamma-1}\rho^{\gamma-1} - \frac{GM}{r} = \text{const.}$$

but $K\gamma\rho^{\gamma-1} = \frac{dP}{d\rho} = c_s^2$

$$\frac{1}{2}v^2 + \frac{c_s^2}{(\gamma-1)} - \frac{GM}{r} = \text{const.}$$

As $r \to \infty$, $v \to 0$; $c_s \to c_s(\infty)$, so, at large r

$$0 + \frac{c_s^2(\infty)}{\gamma-1} - 0 = \text{const.}$$

$$\therefore \quad \frac{1}{2}v^2 + \frac{c_s^2}{\gamma-1} - \frac{GM}{r} = \frac{c_s^2(\infty)}{\gamma-1}$$

At the sonic point, $v^2 = c_s^2$; $r = \frac{GM}{2c_s^2}$, so

$$c_s^2(r_s)\left(\frac{1}{2} + \frac{1}{\gamma-1} - 2\right) = \frac{c_s^2(\infty)}{(\gamma-1)}$$

or $$c_s(r_s) = c_s(\infty)\left(\frac{2}{5-3\gamma}\right)^{1/2}$$

Sound speed @ r_s depends only on conditions at ∞ & γ

Now, since $\dot{M}$ is the same at all radii, evaluate at r_s

$$\dot{M} = 4\pi r_s^2 \rho(r_s) c_s(r_s)$$

From above $c_s^2 = K\gamma\rho^{\gamma-1}$ so

$$\rho(r_s) = \rho(\infty)\left[\frac{c_s(r_s)}{c_s(\infty)}\right]^{2/\gamma-1}$$

Substituting and ASA

$$\dot{M} = \pi G^2 M^2 \frac{\rho(\infty)}{c_s^3(\infty)} \underbrace{\left[\frac{2}{5-3\gamma}\right]^{(5-3\gamma)/2(\gamma-1)}}$$

$\Rightarrow$ Varies from $1 \to \simeq 4.5$ ($\gamma = 5/3$) ($\gamma=1$)

e.g. NS in ISM, $\rho(\infty) \simeq 10^{-24}\,\mathrm{g\,cm^{-3}}$; $c_s \simeq 10\,\frac{\mathrm{km}}{\mathrm{s}}$ $(T \simeq 10^4\,K)$, $\gamma = 1.4$

$$\dot{M} \simeq 1.4\times 10^{11}\left(\frac{M}{M_\odot}\right)^2\left(\frac{\rho(\infty)}{10^{-24}\,\mathrm{g\,cm^{-3}}}\right)\left(\frac{c_s(\infty)}{10\,\frac{\mathrm{km}}{\mathrm{s}}}\right)^{-3}\frac{\mathrm{g}}{\mathrm{s}}$$

$$\rightarrow \quad L_{acc} = \frac{GM\dot{M}}{R_*} \simeq 2\times 10^{31}\,\mathrm{erg/s}$$

To estimate of the capture or accretion radius, r_{acc}, start w/

$$\frac{1}{2}v^2 + \frac{c_s^2}{\gamma-1} - \frac{GM}{r} = \frac{c_s^2(\infty)}{\gamma-1}$$

if $c_s \to c_s(\infty)$, then

$$\frac{1}{2}v^2 - \frac{GM}{r} \simeq 0\text{, so gas is becoming bound}$$