Rather than waiting forever (potentially) to see what the total energy is, look at how much energy is contained in diff. freq. ranges (this is what astronomers do!)

Analyze our time-dependent E-field using Fourier analysis:

Fourier transform: $$\hat{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E(t)\, e^{i\omega t}\, dt$$

Inv. Fourier transform: $$E(t) = \int_{-\infty}^{+\infty} \hat{E}(\omega)\, e^{-i\omega t}\, d\omega$$

(Since $E(t)$ is real, $\hat{E}(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E(t)\, e^{-i\omega t}\, dt = \hat{E}^*(\omega)$ comp. conj.)

Parseval's theorem for F.T.

$$\int_{-\infty}^{\infty} E^2(t)\, dt = 2\pi \int_{-\infty}^{\infty} |\hat{E}(\omega)|^2\, d\omega$$

But $|\hat{E}(\omega)|^2 = \hat{E}(\omega)\hat{E}^*(\omega)$, $|\hat{E}(\omega)|^2 = |\hat{E}(-\omega)|^2$

$$\therefore \int_{-\infty}^{+\infty} E^2(t)\, dt = 4\pi \int_{0}^{\infty} |\hat{E}(\omega)|^2\, d\omega$$

So, the total energy/area in the pulse is

$$\frac{dW}{dA} = c\int_0^{\infty} |\hat{E}(\omega)|^2\, d\omega$$

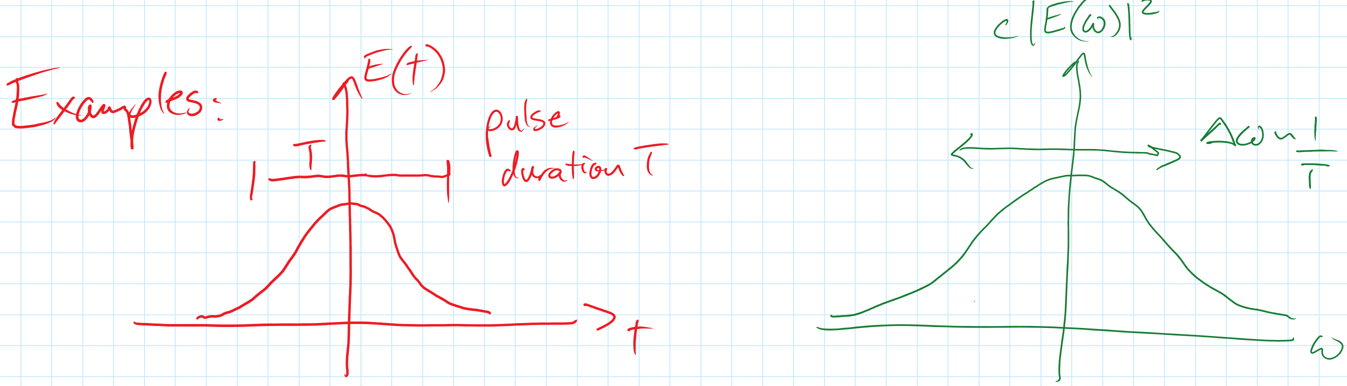
and the energy per unit area per unit freq. in the entire pulse:

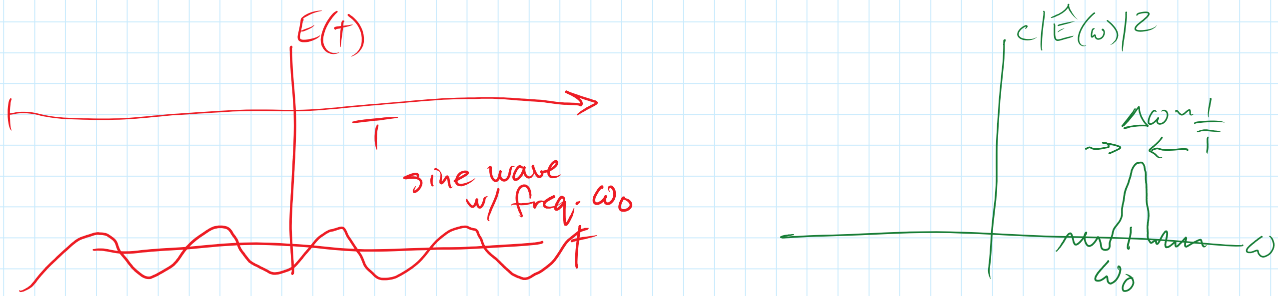
$$\frac{dW}{dA\, d\omega} = c\, |\hat{E}(\omega)|^2$$

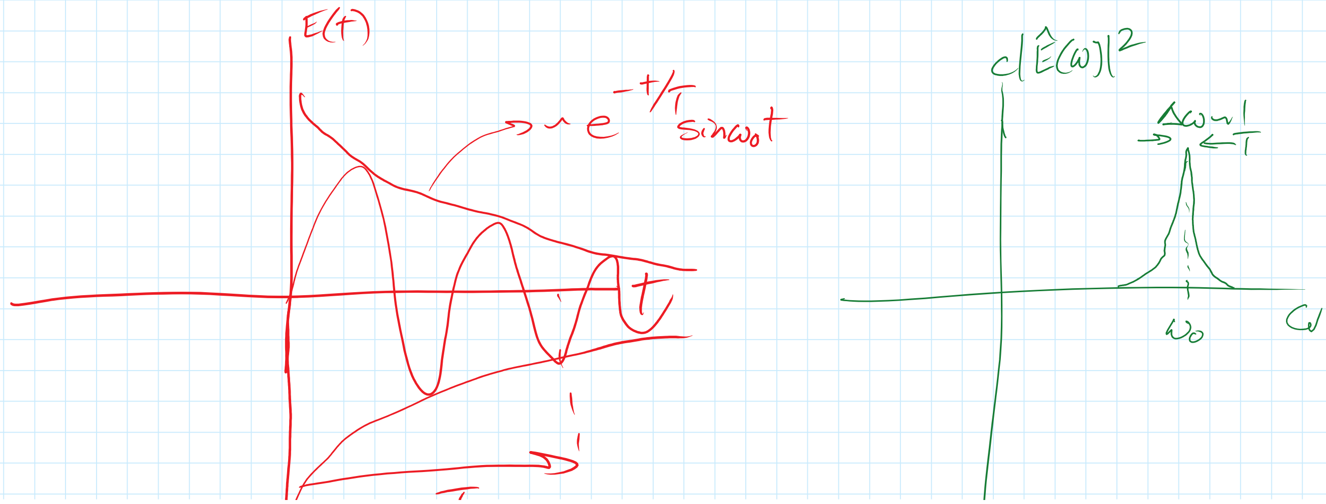
If the pulse repeats on an average timescale T, then formally
$$\frac{dW}{dA\,d\omega\,dt} \equiv \frac{1}{T}\frac{dW}{dA\,d\omega} = \frac{c}{T}|\hat{E}(\omega)|^2$$

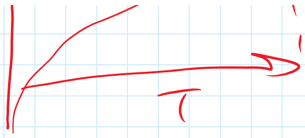
For very long signals that have more or less the same properties over their entire length, then
$$\frac{dW}{dA\,dt\,d\omega} = c \lim_{T\to\infty} \frac{1}{T}|\hat{E}_T(\omega)|^2$$
where $\hat{E}_T(\omega)$ is the transform of a portion of the function $E(t)$ of length T

Examples:









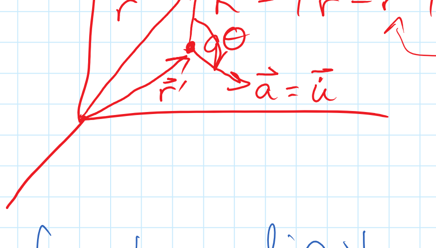
- rules: the extent of the pulse T determines width of finest features in spectrum
  - a sinusoidal dependence within the pulse shape causes the spect. to be concentrated near $\omega \sim \omega_0$

Radiation by Single Charged Particle

As you all know, an accellerated charged particle will radiate. In the radiation zone (far away from the charge), for non-relativistic velocities, the radiated power is given by Larmor's formula:

$$P = \frac{dW}{dt} = \frac{2q^2\dot{u}^2}{3c^3}$$ where $\dot{u}$ is the particle's acc'n evaluated at the retarded time.

(Reminder: $E = B = \frac{q\dot{u}}{Rc^2}\sin\theta$ then calc. S & integrate over a sphere



$\vec{r}'$ is eval at retarded time)

In terms of the dipole moment $\vec{d} = q\vec{r}' \Rightarrow \ddot{\vec{d}} = q\dot{\vec{u}}$

$$\therefore \; P = \frac{2}{3}\frac{|\ddot{\vec{d}}|^2}{c^3}$$

$$\therefore \quad P = \frac{2|\ddot{\vec{d}}|^2}{3c^3}$$

Larmor Radiation Spectrum

The spectrum of the flux averaged over the signal interval T is

$$\frac{dS}{d\omega} = \frac{dP}{dA\,d\omega} = \frac{dW}{dA\,d\omega\,dt} = \frac{c}{T}|\hat{E}(\omega)|^2$$

The total energy emitted per unit bandwidth per unit area is $\displaystyle \frac{dW}{dA\,d\omega} = \frac{dS}{d\omega}T = c|\hat{E}(\omega)|^2$

Consider a time varying or oscillating dipole moment whose magnitude is $d(t)$. Then the E-field transform $\hat{E}(\omega)$ may be written in terms of the transform of $\ddot{d}(t)$, ie $\hat{\ddot{d}}(\omega)$

That is, $\displaystyle E(t) = \frac{q\dot{u}}{Rc^2}\sin\theta = \frac{\ddot{d}(t)\sin\theta}{c^2R}$

The FT of $\displaystyle d(t) = \int_{-\infty}^{+\infty} e^{-i\omega t}\hat{d}(\omega)\,d\omega$

so $\displaystyle \ddot{d}(t) = \underbrace{-\int_{-\infty}^{+\infty}\omega^2\hat{d}(\omega)}_{\text{F.T. of } \ddot{d}(t)} e^{-i\omega t}\,d\omega$

$$\therefore \quad \hat{E}(\omega) = -\frac{1}{c^2R}\omega^2\hat{d}(\omega)\sin\theta$$

$$c^2 R$$

$$\therefore \quad \frac{dW}{dA\,d\omega} = \frac{\omega^4}{c^3 R^2} |\hat{d}(\omega)|^2 \sin^2\theta$$