$\therefore$ The torque/arc length exerted on the outer ring by the inner ring is:

$$\tau/_{\text{arc length}} = \frac{\Delta L}{\text{time arclength}} = H\rho\hat{\nu}\Big[\left(R-\tfrac{\lambda}{2}\right)\left\{\left(R-\tfrac{\lambda}{2}\right)\Omega\left(R-\tfrac{\lambda}{2}\right)-\Omega(R)R\right\}$$
$$-\left(R+\tfrac{\lambda}{2}\right)\left\{\left(R+\tfrac{\lambda}{2}\right)\Omega\left(R+\tfrac{\lambda}{2}\right)-\Omega(R)R\right\}\Big]$$

Keeping terms $O(\lambda)$, we get

$$\tau/_{\text{arc length}} = H\rho\hat{\nu}\Big[R^2\left(\Omega\left(R-\tfrac{\lambda}{2}\right)-\Omega(R)\right)-R^2\left(\Omega\left(R+\tfrac{\lambda}{2}\right)-\Omega(R)\right)$$
$$-R\lambda\left(\Omega\left(R+\tfrac{\lambda}{2}\right)-\Omega(R)+\Omega\left(R-\tfrac{\lambda}{2}\right)-\tfrac{1}{2}\Omega(R)\right)\Big]$$

Since λ is v. small, assume Ω changes slowly over that scale

$$\therefore \frac{\tau}{\text{arc length}} = H\rho\hat{\nu}\left[R^2\frac{d\Omega}{dR}\left(-\frac{\lambda}{2}\right)-R^2\frac{d\Omega}{dR}\left(\frac{\lambda}{2}\right)+O(\lambda^2)\right]$$

$$= -H\rho\hat{\nu}R^2\lambda\frac{d\Omega}{dR}$$

For a circular ring, the torque exerted by the outer ring on the inner ring (= - the torque of the inner on the outer) is

$$G(R) = 2\pi R\nu\Sigma R^2\frac{d\Omega}{dR}$$

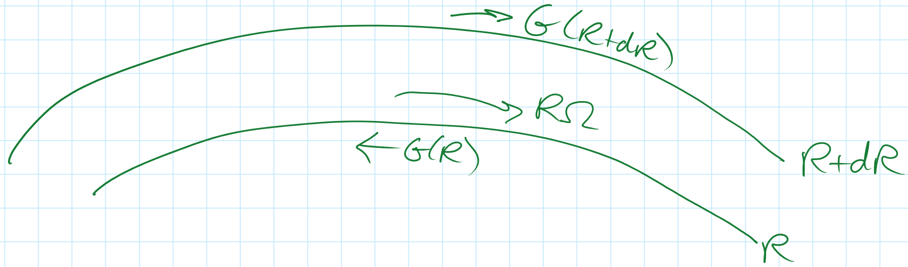
where $\nu = \lambda\hat{\nu}$ is the coefficient of kinetic viscosity

$\Sigma = \rho H$ is the surface density

Note the torque vanishes for rigid rotation.

For a Keplerian rotation law $\frac{d\Omega}{\cdot} < 0$ so $G(R) < 0 \Rightarrow$ the inner ring loses ang. mom. to the dR outer one ; gas spirals in

Lets consider this in detail:



The net torque on this ring (trying to speed it up):

$$G(R+dR) - G(R) = \frac{\partial G}{\partial R} dR$$

Because this torque is acting in the sense of ang. vel. $\Omega(R)$ there is a rate of working by the torque

$$\Omega G_{net} = \Omega \frac{\partial G}{\partial R} dR = \left(\frac{\partial (\Omega G)}{\partial R} - G \frac{\partial \Omega}{\partial R} \right) dR$$

The term $\frac{\partial}{\partial R}(\Omega G)\, dR$ is the rate of 'convection' of rotational energy through the gas by the torque. Its the energy 'passing through'. In contrast, $-G \frac{\partial \Omega}{\partial R} dR$ represents a local rate of loss of mechanical energy to the gas. This lost energy must go into internal energy, ie, heat.

$\therefore$ The viscous torque causes dissipation within the gas at a rate $G \frac{d\Omega}{dR} dR$ per ring width dR

$\therefore$ Dissipation rate per unit ring area $\quad \Omega' = \frac{d\Omega}{dR}$

∴ Dissipation rate per unit ring area $\quad \Omega' = \frac{d\Omega}{dR}$

$$D(R) = \frac{G\Omega' dR}{(2)\, 2\pi R\, dR} = \frac{G\Omega'}{4\pi R} = \frac{\cancel{2\pi R}\, \nu \Sigma R^2 \Omega' \Omega'}{2\cancel{\cdot 4\pi R}}$$

2 faces for each ring

$$= \frac{1}{2} \nu \Sigma (R\Omega')^2$$

$D(R)$ is the energy available to radiate.

Some More Details About Viscosity

The shear viscosity gives a force density in the ϕ direction

$$f_{visc,shear} \sim \rho \lambda \tilde{v} \frac{\partial^2 v_\phi}{\partial R^2} \sim \rho \lambda \tilde{v} \frac{v_\phi}{R^2}$$

The Reynolds number measures the importance of viscosity in a flow:

$$R_e = \frac{\text{inertia}}{\text{viscous}} \sim \frac{v_\phi^2/R}{\lambda \tilde{v} v_\phi / R^2} \sim \frac{v_\phi R}{\lambda \tilde{v}}$$

if $R_e \ll 1$, viscous forces dominate the flow

$R_e \gg 1$, the viscosity associated w/ a given λ ; $\tilde{v}$ is dynamically unimportant

e.g. molecular viscosity,

$$\lambda \sim \lambda_{Debye} = \frac{7\times 10^5}{\ln\Lambda} \frac{T^2}{N} \text{ cm} \quad ; \quad \tilde{v} \simeq c_s = \left(\frac{kT}{\mu m_H}\right)^{-5/2}$$

where N is # density and $\ln\Lambda \simeq 10 + 3.45 \log T - 1.15 \log N_e \simeq 15$

Plugging in you get

$$R_{e;mol} \simeq 0.25\, N \left(\frac{R}{10^{10}\text{cm}}\right)^{1/2} \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{T}{10^4 K}\right)^{-5/2}$$

In a accretion disk, $N > 10^{15}\,cm^{-3}$ so $Re_{mol} > 10^{14}$

$\therefore$ molecular viscosity is too weak

For most fluids, once $Re \gtrsim 10\text{-}10^3$, the flow becomes turbulent in which case the flow will be characterized by a size λ_{turb} & turnover velocity V_{turb}. Since the turbulent motions are chaotic, our simple viscosity calc. still applies w/ $\nu_{turb} \sim \lambda_{turb} V_{turb}$

Clearly $\lambda_{turb} \leq H$ & $V_{turb} \lesssim c_s$, so

$$\nu = \alpha c_s H \quad w/ \ \alpha \leq 1.$$