# Steady-State Thin Disks

Work in Cylindrical coordinates

Disk is at $z=0$, $V_\phi = R\Omega_K(R)$

In addition, the gas is assumed to possess a small radial 'drift' velocity $V_R$ which is negative.

Disk is characterized by $\Sigma(R,t) = \int \rho\, dz$

Continuity eqn: $$R\cancel{\frac{\partial \Sigma}{\partial t}} + \frac{\partial}{\partial R}(R\Sigma V_R) = 0$$ steady-state

Cons. of momentum: $$R\cancel{\frac{\partial}{\partial t}(\Sigma R^2 \Omega)} + \frac{\partial}{\partial R}(V_R R \Sigma R^2 \Omega) = \frac{1}{2\pi}\frac{\partial G}{\partial R}$$ steady-state

As w/ Bondi-Hoyle, mass continuity gives

$R\Sigma V_R = \text{constant}$ and since $V_R < 0$

$$\therefore \dot{M} = 2\pi R\Sigma(-V_R) = \text{const.}$$

Integrate the mom. eqn

$$V_R R \Sigma R^2 \Omega = \frac{1}{2\pi} G + \text{const.}$$

rewrite $$V_R R \Sigma R^2 \Omega = \frac{1}{2\pi} G + \frac{C}{2\pi} \qquad C = \text{const.}$$

Sub. $G = 2\pi R \nu \Sigma R^2 \Omega'$

$$V_R R \Sigma R^2 \Omega = \frac{\cancel{2\pi} R \nu \Sigma R^2 \Omega'}{\cancel{2\pi}} + \frac{C}{2\pi}$$

rearrange $$-\nu\Sigma\Omega' = (-V_R)\Sigma\Omega + \frac{C}{\ldots}$$

To determine $C$, consider the minimum radius of the disk $2\pi R^3$ (ie., just above the NS's surface or ISCO). Call this radius $R^*$.

Inside $R^*$ the gas hits a boundary layer (for a NS) or plunges into the BH. $\therefore \Omega$ is a max at $R^*$ & $\Omega' = 0$.

If $\Omega$ is still close to Keplerian at $R_*$, then

$$\Omega(R_*) = \left(\frac{GM}{R_*^3}\right)^{1/2}$$

then $$0 = -V_R \Sigma \left(\frac{GM}{R_*^3}\right)^{1/2} + \frac{C}{2\pi R_*^3}$$

$$C = -\dot{M}(GMR_*)^{1/2}$$

Sub. this back into mom. eqn & use $\Omega = \left(\frac{GM}{R^3}\right)^{1/2}$ and

$$\Omega' = -\frac{3}{2}\left(\frac{GM}{R^5}\right)^{1/2}$$

$$\therefore \quad \nu\Sigma = \frac{\dot{M}}{3\pi}\left(1 - \left(\frac{R_*}{R}\right)^{1/2}\right)$$

Recall the viscous dissipation per unit area, $D(R) = \frac{1}{2}\nu\Sigma R^2 \Omega'^2$

$$\boxed{\therefore \; D(R) = \frac{3GM\dot{M}}{8\pi R^3}\left(1 - \left(\frac{R_*}{R}\right)^{1/2}\right)}$$

- independent of viscosity (assumes viscosity can adjust to give the same $\dot{M}$ everywhere)

Luminosity produced by disk b/w $R_1$ & $R_2$

$$L(R_1, R_2) = 2\int^{R_2} D(R)\, 2\pi R\, dR = \frac{3GM\dot{M}}{2}\int^{R_2}\left(1 - \left(\frac{R_*}{R}\right)^{1/2}\right)\frac{dR}{R^2}$$

$$L(R_1,R_2) = 2\int_{R_1} D(R)\, 2\pi R\, dR = \frac{3GM\dot{M}}{2}\int_{R_1}\left(1-\left(\frac{R_*}{R}\right)^{1/2}\right)\frac{dR}{R^2}$$

evaluate this integral by setting $y = \frac{R_*}{R}$

$$L(R_1,R_2) = \frac{3GM\dot{M}}{2}\left[\frac{1}{R_1}\left(1-\frac{2}{3}\left(\frac{R_*}{R_1}\right)^{1/2}\right) - \frac{1}{R_2}\left(1-\frac{2}{3}\left(\frac{R_*}{R_2}\right)^{1/2}\right)\right]$$

Let $R_1 = R_*$ and let $R_2 \to \infty$ get the luminosity of the whole disk $L_{disk} = \frac{GM\dot{M}}{2R_*} = \frac{1}{2}L_{acc}$

$\frac{1}{2}$ of the grav. P.E. is radiated away / $\frac{1}{2}$ is retained as KE
is released for NSs

The total rate at which energy is dissipated in a ring b/w $R$ & $R+dR$ is

$$2 \times 2\pi R\, dR\, D(R) = \frac{3GM\dot{M}}{2R^2}\left(1-\left(\frac{R_*}{R}\right)^{1/2}\right) dR$$

Of this total, $\frac{1}{2}\frac{GM\dot{M}}{R^2}dR$ comes from the rate of release of grav. p.e. b/w $R$ & $R+dR$ ($\frac{1}{2}$ of this goes into K.E.)

The remainder $= \frac{GM\dot{M}}{R^2}\left(1-\frac{3}{2}\left(\frac{R_*}{R}\right)^{1/2}\right)dR$ is the flow of energy into the annulus associated w/ the transport of ang. mom. outwards.

Indeed at $R > \frac{9R_*}{4}$, the rate of energy release from ang. mom. transport > release of local binding energy

The upshot is the disk is brighter over a wider range of radii due to ang. mom. transport.

Condition for a thin Disk:

Consider the structure in z-direction. As there is no flow in this direction, hydrostatic balance

$$\frac{1}{\rho}\vec{\nabla}P = -\vec{\nabla}\Phi$$

$$\frac{1}{\rho}\frac{dP}{dz} = -\frac{\partial}{\partial z}\left[\frac{GM}{(R^2+z^2)^{1/2}}\right]$$

For a thin disk, $z \ll R$, so

$$\frac{1}{\rho}\frac{dP}{dz} = \frac{1}{2}\frac{GM}{(R^2+z^2)^{3/2}}2z = \frac{GMz}{\left(z^2\left(\frac{R^2}{z^2}+1\right)\right)^{3/2}} \simeq \frac{GMz}{R^3}$$

If the typical pressure scale height is $z \simeq H$ then

$$\frac{dP}{dz} \sim \frac{P}{H} \quad \text{so} \quad \frac{1}{\rho}\frac{P}{H} \sim \frac{GMH}{R^3}$$

But $\frac{P}{\rho} \sim c_s^2$, so $c_s^2 \sim \frac{GMH^2}{R^3} \sim \frac{GM}{R}\left(\frac{H}{R}\right)^2$

$$\Rightarrow \left(\frac{H}{R}\right) \simeq c_s \Big/ \sqrt{\frac{GM}{R}} = c_s / v_{Kepl.}$$

So, for $\frac{H}{R} \ll 1$, $c_s \ll v_K$

i.e., for a thin disk, we require that the local Kepler velocity be highly supersonic. This is a condition on the temp. of the disk and therefore the cooling mechanism.