Check if $V_\phi \sim V_{Kepler}$ for a thin disk

Radial comp. of Euler eqn.

$$V_R \frac{\partial V_R}{\partial R} - \frac{V_\phi^2}{R} + \frac{1}{\rho}\frac{\partial P}{\partial R} + \frac{GM}{R^2} = 0$$

$$V_R = \frac{-\dot{M}}{2\pi R \Sigma} = \frac{-\dot{M}}{2\pi R\left(\frac{\dot{M}}{3\pi\nu}\left(1-\left(\frac{R_*}{R}\right)^{1/2}\right)\right)} = -\frac{3\nu}{2R\left(1-\left(\frac{R_*}{R}\right)^{1/2}\right)}$$

so, $V_R \sim \frac{\nu}{R} \sim \frac{\alpha c_s H}{R} \ll c_s$ for a thin disk since $\frac{H}{R} \ll 1$

but $\frac{1}{\rho}\frac{\partial P}{\partial R} \sim \frac{c_s^2}{R}$, so radial velocity term is going to be much smaller than this press. grad.

$$\therefore \frac{V_\phi^2}{R} = \frac{GM}{R^2} + \frac{1}{\rho}\frac{\partial P}{\partial R} \simeq \frac{GM}{R^2} + \frac{c_s^2}{R}$$

$$\text{or} \quad V_\phi^2 = \frac{GM}{R} + c_s^2 = \frac{GM}{R}\left(1 + c_s^2 \Big/ \left(\frac{GM}{R}\right)\right)$$

$$= \frac{GM}{R}\left(1 + O(\mathcal{M}^{-2})\right)$$

where $\mathcal{M} = \frac{V_K}{c_s}$ is the Mach #

Thus, in a thin disk,

V_ϕ is Keplerian & highly supersonic. V_R & H are self-consistently small. $H \simeq c_s\left(\frac{R}{GM}\right)^{1/2} R = \frac{c_s}{V_\phi} R$

The Local Structure of Thin Disks

The radial movement is small. Know the radial energy generation

rate $D(R)$. The temp. & pressure gradients are vertical.
So, can treat disk as a 1D stellar atmosphere problem.

For an isothermal z-structure, hydrostatic equilibrium:

$$\rho(R) = \rho_c(R) e^{-z^2/2H^2}$$

Define an approx. disk density by

$$\Sigma = \int_0^\infty \rho(z)\,dz = \rho_c \int_0^\infty e^{-z^2/2H^2} dz \simeq \rho_c H \quad \text{since } e^{-z^2/2H^2} \text{ falls rapidly}$$

$$\therefore \rho_c \simeq \frac{\Sigma}{H} \quad \& \quad H = \frac{c_s}{v_\phi} R$$

The sound speed is $c_s^2 = \frac{P}{\rho}$

where $P = P_{gas} + P_{rad} = \frac{\rho k T_c}{\mu m_p} + \frac{4\sigma}{3c} T_c^4$

σ = Stefan-Boltzmann constant (from $L = 4\pi R^2 \sigma T^4$)

& we have assumed $T(R,z) \simeq T(R,0) = T_c$

The disk is assumed to be optically thick in the z-direction.

$\Upsilon = \rho H \kappa = \Sigma \kappa >> 1$ where κ is the opacity/g (cm^2/g).

Then the vertical energy flux transport is radiative/diffusive

$$F(z) = -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z} = \frac{-16\sigma T^3 H}{3\Upsilon}\frac{\partial T}{\partial z} = -\frac{4\sigma H}{3\Upsilon}\frac{\partial T^4}{\partial z} \sim -\frac{4\sigma T^4(z)}{3\Upsilon}$$

But $D(R) = F(H) - F(0) \simeq \frac{4\sigma T_c^4}{3\Upsilon}$ so long as $T_c^4 >> T^4(H)$

So, the set of 8 equations that describe a s.s. accretion disk:

1. $\rho = \Sigma$ ... 5. $4\sigma T_c^4 = 3GM\dot{M}$ ...

1. $\rho = \frac{\Sigma}{H}$

2. $H = \frac{c_s R^{3/2}}{(GM)^{1/2}}$

3. $c_s^2 = \frac{P}{\rho}$

4. $P = \frac{\rho k T_c}{\mu m_p} + \frac{4\sigma T_c^4}{3c}$

5. $\frac{4\sigma T_c^4}{3\tau} = \frac{3GM\dot{M}}{8\pi R^3}\left(1 - \left(\frac{R_*}{R}\right)^{1/2}\right)$

6. $\tau = \Sigma \kappa$ ← needs to be specified

7. $\nu \Sigma = \frac{\dot{M}}{3\pi}\left(1 - \left(\frac{R_*}{R}\right)^{1/2}\right)$

8. $\nu = \nu(\rho, T, \Sigma, \alpha)$ ← must be specified

Solving gives 8 eqn's for ρ, H, Σ, c_s, P, T_c, τ & ν as functions of M, $\dot{M}$, R & α. (v_R also follows from $v_R = \frac{-\dot{M}}{2\pi R \Sigma}$)

Consequence of the assumption that the disk is optically thick in the z-direction. If this holds at all R, then

$$\sigma T^4(R) = D(R)$$

For $R > R_*$

verify this → $T(R) = T_*\left(\frac{R}{R_*}\right)^{-3/4}$ where $T_* = \left(\frac{3GM\dot{M}}{8\pi\sigma R_*^3}\right)^{1/4}$

For a Schw. BH w/ $R_* = 6\frac{GM}{c^2}$,

$$T_* = (2.74 \times 10^7\,K)\left(\frac{\dot{M}}{\dot{M}_{Edd}}\right)^{1/4}\left(\frac{\eta}{0.1}\right)^{-1/4}\left(\frac{M}{M_\odot}\right)^{-1/4}$$ verify this

The max of $T(R)$ is at $R = \frac{49}{36}R_*$ and $T_{max} = 0.488\,T_*$