We had $\dfrac{dW}{dA\,d\omega} = \dfrac{1}{c^3}\dfrac{\omega^4}{R^2}|\hat{d}(\omega)|^2 \sin^2\theta$

To get the total energy per freq. interval, integrate over the surface of sphere R

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\hat{d}(\omega)|^2 \quad \leftarrow \text{Only for non-rel. particles}$$

$\therefore$ The spectrum of dipole radiation is $\propto$ the freq. of the oscillation of the dipole moment.

Relativistic Larmor Formula

The emitted power is invariant under Lorentz transformations: $P = \dfrac{dW}{dt}$ (lab frame) $= \dfrac{dW'}{dt'} = P'$ (rest frame)

albeit momentarily and moving non-relativictically

So, $P = P' = \dfrac{2q^2}{3c^3}|\vec{a}'|^2$

It can be shown that $a'_{\parallel} = \gamma^3 a_{\parallel}$ and $a'_{\perp} = \gamma^2 a_{\perp}$

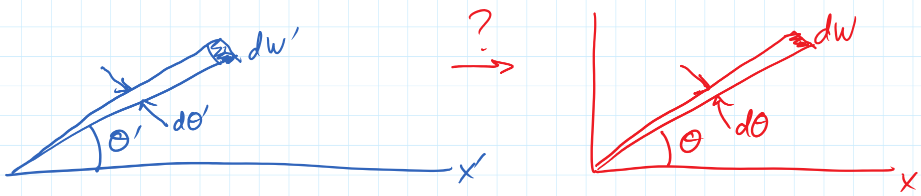
$\left(\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)$

$$P = \frac{2q^2\gamma^4}{3c^3}\left(a_{\perp}^2 + \gamma^2 a_{\parallel}^2\right)$$

A small acc'n of a rel. particle leads to a large energy loss!

Angular Dist'n of Emitted & Received Power

Consider an amount of energy dW' that is emitted into the solid angle $d\Omega' = \sin\theta'\,d\theta'\,d\phi'$ about the direction at angle θ' to the x'-axis



let $\mu = \cos\theta$ & $\mu' = \cos\theta'$

so that $d\Omega = d\mu\, d\phi$ & $d\Omega' = d\mu' d\phi'$

Recall that energy & momentum form a four-vector so the transformation of the energy of the radiation is

$$dW = \gamma(dW' + v\, dp_x') = \gamma\left(1 + v\frac{dp_x'}{dW'}\right)dW' = \gamma\left(1 + \frac{v}{c}c\frac{dp_x'}{dW'}\right)dW'$$

$$dW' = dp'c \rightarrow dp_x' = \cos\theta' dp' = \cos\theta' \frac{dW'}{c} = \mu' \frac{dW'}{c}$$

$$dW = \gamma(1+\beta\mu')dW' \qquad \left(\beta = \frac{v}{c}\right)$$

From the aberration of light,

$$\mu = \frac{\mu' + \beta}{1+\beta\mu'}$$

differentiating, $\dfrac{d\mu}{d\mu'} = (1+\beta\mu')^{-1} + (\mu'+\beta)(-1)(1+\beta\mu')^{-2}\beta$

$$= \frac{1 + \cancel{\beta\mu'} - \cancel{\beta\mu'} - \beta^2}{(1+\beta\mu')^2} = \frac{1-\beta^2}{(1+\beta\mu')^2}$$

but $\gamma = \dfrac{1}{\sqrt{1-\beta^2}} \rightarrow 1-\beta^2 = \dfrac{1}{\gamma^2}$

$$\therefore d\mu = \frac{d\mu'}{\gamma^2(1+\beta\mu')^2}$$

and, since $d\phi = d\phi'$

$$d\Omega = d\mu\, d\phi = \frac{d\mu' d\phi'}{\gamma^2(1+\beta\mu')^2} = \frac{d\Omega'}{\gamma^2(1+\beta\mu')^2}$$

Thus, the radiated energy per unit solid angle in the lab frame is $\underline{dW} = \gamma(1+\beta\mu')dW' \left(\gamma^2(1+\beta\mu')^2\right)$

Thus, the radiated energy per unit solid angle in the lab frame is $\frac{dW}{d\Omega} = \frac{\gamma(1+\beta\mu')dW'}{d\Omega'}\left(\gamma^2(1+\beta\mu')^2\right)$

$$= \gamma^3(1+\beta\mu')^3\frac{dW'}{d\Omega'}$$

The ang. dist'n of the rest frame power is found by dividing $\frac{dW'}{d\Omega'}$ by dt'. In the lab-frame the received power requires including time-dilation & retardation effects

$$\text{ie, } dt = \gamma(1-\beta\mu)dt' = \gamma\left(1-\beta\left[\frac{\mu'+\beta}{1+\beta\mu'}\right]\right)dt'$$

$$= \frac{dt'}{\gamma(1+\beta\mu')}$$

$$\therefore \frac{dP}{d\Omega} = \frac{dW}{d\Omega dt} = \frac{\gamma^3(1+\beta\mu')^3}{dt'}\frac{dW'}{d\Omega'}\gamma(1+\beta\mu')$$

$$\text{or} \quad \frac{dP}{d\Omega} = \gamma^4(1+\beta\mu')^4\frac{dP'}{d\Omega'}$$

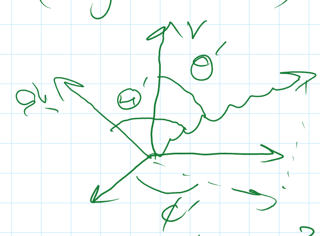
To get rid of μ' on RHS, use $\mu = \frac{\mu'+\beta}{1+\beta\mu'}$ & flip signs and primes $\mu' = \frac{\mu-\beta}{1-\beta\mu}$

ASA...

$$\frac{dP}{d\Omega} = \frac{1}{\gamma^4(1-\beta\mu)^4}\frac{dP'}{d\Omega'}$$

In the instantaneous rest frame of the particle, the ang. dist'n is given by a dipole (eg. Larmor)

$$\frac{dP'}{d\Omega'} = \frac{q^2 a'^2 \sin^2\Theta'}{4\pi c^3}$$



Writing $\vec{a}' = \vec{a}'_\parallel + \vec{a}'_\perp$ and using $a'_\parallel = \gamma^3 a_\parallel$ & $a'_\perp = \gamma^2 a_\perp$

We get $\boxed{\frac{dP}{} = \frac{q^2}{}\left(\gamma^2 a_\parallel^2 + a_\perp^2\right)\sin^2\Theta'}$

we get $\boxed{\dfrac{dP}{d\Omega} = \dfrac{q^2}{4\pi c^3} \dfrac{(\gamma^2 a_{\parallel}^2 + a_{\perp}^2)}{(1-\beta\mu)^4} \sin^2 \psi'}$