High-Energy Astrophysics
Problem Set 1 — Solutions

1. (a) The Lorentz transformation for differentials are $dx = \gamma(dx' + v dt')$, $dy = dy'$, $dz = dz'$, and $dt = \gamma(dt' + (v/c^2)dx') = \gamma\sigma dt'$, where

$$\sigma \equiv 1 + \frac{vu'_x}{c^2}.$$

Combine these with the Lorentz transformations of velocities:

$$\begin{aligned} u_x &= \frac{dx}{dt} = \frac{\gamma(dx' + v dt')}{\gamma(dt' + (v/c^2)dx')} = \frac{u'_x + v}{1 + vu'_x/c^2}, \\ u_y &= \frac{u'_y}{\gamma(1 + vu'_x/c^2)}, \\ u_z &= \frac{u'_z}{\gamma(1 + vu'_x/c^2)} \end{aligned}$$

to compute changes in velocities measured in different frames:

$$\begin{aligned} du_x &= \gamma^{-2}\sigma^{-2} du'_x, \\ du_y &= \gamma^{-1}\sigma^{-2}\left(\sigma du'_y - \frac{vu'_y}{c^2} du'_x\right). \end{aligned}$$

Hence,

$$\begin{aligned} a_x &= \frac{du_x}{dt} = \gamma^{-3}\sigma^{-3}\frac{du'_x}{dt'} = \gamma^{-3}\sigma^{-3} a'_x, \\ a_y &= \frac{du_y}{dt} = \gamma^{-2}\sigma^{-3}\left(\sigma\frac{du'_y}{dt'} - \frac{vu'_y}{c^2}\frac{du'_x}{dt'}\right), \\ &= \gamma^{-2}\sigma^{-3}\left(\sigma a'_y - \frac{vu'_y}{c^2} a'_x\right). \end{aligned}$$

A similar result holds for a_z.

(b) If the particle is at rest instantaneously in K', then $u'_x = u'_y = u'_z = 0$. Then $\sigma = 1$, and from part (a),

$$\begin{aligned} a'_\parallel &= \gamma^3 a_\parallel, \\ a'_\perp &= \gamma^2 a_\perp. \end{aligned}$$

2. The knee in the spectrum gives T:

$$T = \frac{E_{\rm max}}{k} \approx 10^9 \text{ K}.$$

In class, we showed that the total emissivity from a bremsstahlung plasma is

$$\epsilon_{\rm bremss} = (1.4 \times 10^{-27}) T^{1/2} n_e n_i Z^2 \bar{g_B} \text{ erg s}^{-1} \text{ cm}^{-3},$$

so the observed flux will be

$$F = \frac{1}{4\pi L^2} \frac{4\pi R^3}{3} (1.4 \times 10^{-27} T^{1/2} n_e n_i Z^2 \bar{g_B}).$$

At $T = 10^9$ K the gas is completely ionized. If we can assume it is pure hydrogen, $n_i = n_e$. (Including a typical He abundance makes only a negligible difference.) Then

$$n_i n_e \approx n_{\rm H}^2 = \left(\frac{\rho}{m_{\rm H}}\right)^2 = 3.6 \times 10^{47} \rho^2.$$

Taking $Z = 1$ and $\bar{g_B} = 1.2$ gives

$$F = 2.0 \times 10^{20} \rho^2 T^{1/2} R^3 L^{-2}.$$

Hydrostatic equilibrium gives another constraint on ρ and R. From the virial theorem we known that 2×(kinetic energy/particle)= −(gravitational energy/particle) or

$$3kT \sim \frac{GMm_{\rm H}}{R}.$$

For $T = 10^9$ K this implies

$$R \approx 5 \times 10^8 \left(\frac{M}{M_\odot}\right) \text{ cm}.$$

Combining this expression for R with the equation for F gives the following constraint on ρ:

$$\rho \approx 4 \times 10^{-26} L F^{1/2} \left(\frac{M}{M_\odot}\right)^{-3/2}.$$

Substituting in the measured values of F and L we obtain

$$\rho \approx 1.2 \times 10^{-7} \left(\frac{M}{M_\odot}\right)^{-3/2} \text{ g cm}^{-3}.$$

The optical depth of a free-free emitting plasma at a frequency ν is

$$\tau_\nu = (8.235 \times 10^{-2}) T^{-1.35} \left(\frac{\nu}{\text{GHz}}\right)^{-2.1} E_M a(\nu, T),$$

where

$$E_M = Z^2 \int n_e n_i ds$$

is the emisssion measure in pc cm^{-6}. Taking $a \approx 1$, and substituting the above results we find

$$\tau_\nu \approx (4.9 \times 10^{10}) \left(\frac{\nu}{\text{GHz}}\right)^{-2.1} \left(\frac{M}{M_\odot}\right)^{-2}$$

at 1 keV the optical depth is

$$\tau_{1\text{ keV}} \approx 1.2 \times 10^{-7} \left(\frac{M}{M_\odot}\right)^{-2}$$

So for the emission to be optically thin at 1 keV, $\tau_{1\text{ keV}} \ll 1$, or

$$\left(\frac{M}{M_\odot}\right) \gg 3.5 \times 10^{-4}$$

To check if electron scattering plays a role, calculate the optical depth to electron scattering,

$$\tau_{\text{es}} \approx \kappa_e \rho R,$$

where $\kappa_e = 0.4$ cm^2 g^{-1}. From the above results,

$$\tau_{\text{es}} \approx 24 \left(\frac{M}{M_\odot}\right)^{-1/2}.$$

Thus, electron scattering will be important unless $\tau_{\text{es}} \ll 1$ or

$$\left(\frac{M}{M_\odot}\right) \gg 576.$$

3. (a)

$$\begin{aligned}\vec{E} &\propto \hat{n}\times(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}} = \hat{n}\times\hat{n}\times\dot{\vec{\beta}} - \hat{n}\times\vec{\beta}\times\dot{\vec{\beta}} \\ &= \hat{n}(\dot{\vec{\beta}}\cdot\hat{n}) - \dot{\vec{\beta}}(\hat{n}\cdot\hat{n}) - \vec{\beta}(\dot{\vec{\beta}}\cdot\hat{n}) + \dot{\vec{\beta}}(\hat{n}\cdot\vec{\beta})\end{aligned}$$

The observer lies at the pulse center, i.e., $\hat{n}$ and $\vec{\beta}$ are parallel at the peak of the pulse. But, for synchrotron radiation, $\vec{\beta}\perp\dot{\vec{\beta}}$, so $\hat{n}\perp\dot{\vec{\beta}}$ at the peak of the pulse, i.e., $\dot{\vec{\beta}}\cdot\hat{n}=0$.

So, from above, $\vec{E}\propto\dot{\vec{\beta}}((\hat{n}\cdot\vec{\beta})-(\hat{n}\cdot\hat{n}))$.

That is, $\vec{E}$ lies along $\dot{\vec{\beta}}$ for synchrotron radiation. Thus, synchrotron radiation is coherent and polarized with a polarization vector that is proportional to the acceleration, i.e., the magnetic field $\vec{B}$, although in general it is proportional to the projected magnetic field. Thus, if the intrinsic polarization vector can be measured, the intrinsic projected magnetic field can be derived.

(b) (i) With non-relativistic electrons ($\beta \ll 1$), $\vec{E}\propto\hat{n}\times(\hat{n}\times\dot{\vec{\beta}}) = \hat{n}(\dot{\vec{\beta}}\cdot\hat{n}) - \dot{\vec{\beta}}(\hat{n}\cdot\hat{n})$.

Assume $\hat{n}\|\vec{B}\rightarrow\hat{n}\perp\dot{\vec{\beta}}\rightarrow\dot{\vec{\beta}}\cdot\hat{n}=0$.

Therefore, $\vec{E}\propto-\dot{\vec{\beta}}|\hat{n}|^2$. So, $\vec{E}$ lies along $\dot{\vec{\beta}}$ for non-relativistic electrons.

(ii) With partially relativistic electrons ($\beta\sim 1/2$), go back to

$$\vec{E}\propto\hat{n}(\dot{\vec{\beta}}\cdot\hat{n}) - \dot{\vec{\beta}}(\hat{n}\cdot\hat{n}) - \vec{\beta}(\dot{\vec{\beta}}\cdot\hat{n}) + \dot{\vec{\beta}}(\hat{n}\cdot\vec{\beta})$$

Again, $\hat{n}\|\vec{B}\rightarrow\hat{n}\perp\dot{\vec{\beta}}\rightarrow\dot{\vec{\beta}}\cdot\hat{n}=0$. In addition, the pitch angle is 90°, i.e., $\vec{\beta}\perp\vec{B}\rightarrow\vec{\beta}\perp\hat{n}\rightarrow\beta\cdot\hat{n}=0$.

Therefore, $\vec{E}\propto-\dot{\vec{\beta}}|\hat{n}|^2$. So, $\vec{E}$ lies along $\dot{\vec{\beta}}$ for partially relativistic electrons.

(c) The Larmor power for synchrotron radiation is

$$P=\frac{2q^2\gamma^4a_\perp^2}{3c^3}.$$

The acceleration from the magnetic field is

$$a_\perp=\frac{qvB\sin\theta}{\gamma mc},$$

where θ is the pitch angle.

Therefore,

$$P=\frac{2q^4\gamma^2v^2B^2\sin^2\theta}{3c^5m^2}.$$

Now, $\gamma = 1/\sqrt{1 - v^2/c^2}$, or $v^2/c^2 = (\gamma^2 - 1)/\gamma^2$. Therefore, the power emitted by an arbitrary synchrotron electron is

$$P = \frac{2q^4 B^2 \sin^2\theta(\gamma^2 - 1)}{3c^3 m^2},$$

or, if the dependent variable is v/c,

$$P = \frac{2q^4 B^2 \sin^2\theta}{3c^3 m^2} \frac{(v/c)^2}{(1 - (v/c)^2)}.$$

When $v/c \ll 1$,

$$P = \frac{2q^4 B^2 \sin^2\theta v^2}{3c^5 m^2}.$$

In the non-relativistic limit the elctron kinetic energy is $E = \frac{1}{2}mv^2$, therefore

$$P = \frac{4q^4 B^2 \sin^2\theta E}{3c^5 m^3}.$$

So, $P \propto E$ in the non-relativistic limit, but $P \propto E^2$ in the ultra-relativistic limit. That is, the dependence of the synchrotron power on the elctron energy gets stronger as the energy increases. This is all due to the factor $\gamma^2 - 1$ which goes like v^2 at low energy ($\gamma \sim 1$) but like E^2 at high energy ($\gamma \gg 1$).

4. Assume electrons with a power-law distribution: $N(E)dE = KE^{-p}dE$. Subsitituing into the expression for α_ν gives

$$\alpha_\nu = \frac{c^2 K(p+2)}{8\pi\nu^2} \int_0^\infty P(\nu, E) E^{-p-1} dE,$$

Assume all the power gets radiated at the critical frequency ν_c. Therefore,

$$\alpha_\nu = \frac{c^2 K(p+2)}{8\pi\nu^2} P(E) E^{-p-1} \frac{dE}{d\nu},$$

where

$$P(E) = \frac{2q^4 B_\perp^2}{3m^2 c^3} \left(\frac{E}{mc^2}\right)^2$$

and

$$\nu = \nu_c = \frac{3}{4\pi} \left(\frac{qB_\perp}{\gamma mc}\right) \left(\frac{E}{mc^2}\right)^3$$

From these expressions, one can calculate $dE/d\nu$ and $E(\nu_c)$. Substituting and reducing results in

$$\alpha_\nu = (p+2)\left(\frac{q^3}{18m}\right)\left(\frac{3q}{4\pi m^3 c^5}\right)^{p/2} KB_\perp^{(p+2/2)}\nu^{-(p+4/2)}.$$

Like our expression for the emissivity, this expression for α_ν has the same dependence on the magnetic field and frequency as the exact expression.

5. The relation between the photon's scattered energy ϵ_s and incident energy ϵ is

$$\epsilon_s = \gamma^2\epsilon\left(1+\frac{\gamma\epsilon}{mc^2}\right)^{-1}\left(1-\frac{v}{c}\cos\theta\right)\left(1+\frac{v}{c}\cos\theta_s'\right)$$

Consider soft radiation so $\gamma\epsilon \ll mc^2$, i.e., $\gamma\epsilon/mc^2 \ll 1$. Therefore,

$$\epsilon_s \approx \gamma^2\epsilon\left(1-\frac{v}{c}\cos\theta\right)\left(1+\frac{v}{c}\cos\theta_s'\right)$$

Also we will assume $(1-v/c) = 1/(2\gamma^2)$ and $(1+v/c) = 2$.

(a) $\theta = 0$, $\theta_s = \theta_s' = \theta_a$
($\theta_a = \langle\theta\rangle = \pi/2 \longrightarrow \langle\cos\theta\rangle = \cos\theta_a = 0$) From above,

$$\epsilon_s \approx \frac{\epsilon}{2}$$

The scattered photon has one-half of its incident energy.

(b) $\theta = 0$, $\theta_s = \theta_s' = \pi$
In this case,

$$\epsilon_s \approx \frac{\epsilon}{4\gamma^2}$$

Since $\gamma \gg 1$, the scattered energy will be very small.

(c) $\theta = \theta_a$, $\theta_s = \theta_s' = \pi$
From above,

$$\epsilon_s \approx \frac{\epsilon}{2}$$

6. Inverse Compton is important if the Compton y parameter exceeds unity:

$$y = \left(\frac{4kT}{mc^2}\right)\tau^2 \gg 1$$

The optical depth is $\tau_{es} \sim \kappa_{es}\rho R$. From the solution to Problem 2,

$$T \approx 10^9 \text{ K}.$$

$$R \approx 5 \times 10^8 \left(\frac{M}{M_\odot} \right) \text{ cm.}$$

$$\rho \approx 1.2 \times 10^{-7} \left(\frac{M}{M_\odot} \right)^{-3/2} \text{ g cm}^{-3}.$$

Thus,

$$y \sim 400 \left(\frac{M}{M_\odot} \right)^{-1}$$

If $M \gg 400\ \mathrm{M}_\odot$, inverse Compton can be ignored, and the determination of T, ρ and R on the assumption of pure bremsstrahlung cooling is self-consistent. On the other hand, if $M < 400\ \mathrm{M}_\odot$, then the model is self-inconsistent, because inverse Compton cooling was ignored in determining the energy balance.

7. i. The characteristic synchrotron frequency is

$$\nu = \nu_c = \frac{3}{4\pi} \left(\frac{qB_\perp}{\gamma mc} \right) \left(\frac{E}{mc^2} \right)^3$$

Taking a pitch angle of $\sin\theta = 3^{-1/2}$,

$$h\nu_c \approx 0.10 \text{ eV} \left(\frac{\gamma}{10^4} \right)^2 \left(\frac{B}{0.1 \text{ G}} \right).$$

The ratio of the photon's energy to the electron rest mass energy, in the electron rest frame, is the given approximately by

$$\frac{\gamma h\nu_c}{mc^2} \approx 2.0 \times 10^{-3} \left(\frac{\gamma}{10^4} \right)^3 \left(\frac{B}{0.1 \text{ G}} \right).$$

ii. The energy associated with a temperature of 1 K is $\sim 0.86 \times 10^{-4}$ eV. The blackbody spectrum peaks at $\sim 2.8kT$. Thus, the characteristic photon in a blackbody spectrum of temperature T has an energy $\sim 2.4 \times 10^{-4} T$ eV. The ratio of a microwave photon energy to electron rest mass in the latter's rest frame is, therefore,

$$\frac{\gamma h\nu_c}{mc^2} \approx 1.4 \times 10^{-5} \left(\frac{\gamma}{10^4} \right).$$